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Master 1 Maths
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Finish Exam on Spectrum Theory course

Correction

Exercise 1.
1) Since S, T are self–adjoints then, $\mathcal{A}^* = S^* - iT^* = S - iT$.
By addition and subtraction of $\mathcal{A} = S + iT$ and $\mathcal{A}^* = S - iT$, we get

$$\mathcal{A} + \mathcal{A}^* = 2S, \quad \mathcal{A} - \mathcal{A}^* = 2iT.$$

Thus,

$$S = \frac{\mathcal{A} + \mathcal{A}^*}{2}, \quad T = \frac{\mathcal{A} - \mathcal{A}^*}{2i}.$$

2) We have

$$\mathcal{A}\mathcal{A}^* - \mathcal{A}^*\mathcal{A} = (S + iT)(S - iT) - (S - iT)(S + iT) = -2i(ST - TS).$$

Thus, $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A} \Leftrightarrow ST = TS$, then $\mathcal{A}$ is normal if and only if $ST = TS$.
3) By substitution we get $\mathcal{A}\mathcal{A}^* = (S + iT)(S - iT) = S^2 - iST + iTS + T^2$, therefore,

$$\mathcal{A}\mathcal{A}^* = S^2 + T^2.$$

Suppose that $\mathcal{A}$ invertible, then $\mathcal{A}^*$ is invertible, so is $S^2 + T^2 = \mathcal{A}\mathcal{A}^*$.
Conversely, if $S^2 + T^2$ is invertible, then, multiplying by $(S^2 + T^2)^{-1}$, we get

$$\mathcal{A}\left(\mathcal{A}^*(S^2 + T^2)^{-1}\right) = \left((S^2 + T^2)^{-1}\mathcal{A}^*\right)\mathcal{A} = I,$$

which shows that $\mathcal{A}$ is invertible and

$$\mathcal{A}^{-1} = \mathcal{A}^*(S^2 + T^2)^{-1}.$$

Exercise 2.
i) Since $t \in [0, x]$ then $|x - t| = x - t$. Hence,

$$I = \int_0^x |x - t|^2 dt = \int_0^x (x - t)^2 dt = \left.\frac{-(x - t)^3}{3}\right|_0^x = \frac{x^3}{3}.$$

ii)

$$\|\mathcal{T}f\|^2 = \int_0^1 |\mathcal{T}f(x)|^2 dx = \int_0^1 \left|\left(\int_0^x (x - t)f(t)dt\right)\right|^2 dx$$

using Cauchy Schwarz inequality on the interior integral, we get

$$\|\mathcal{T}f\|^2 \le \int_0^1 \left(\int_0^x |x - t|^2 dt\right)\left(\int_0^x |f(t)|^2 dt\right) dx$$

$$\|\mathcal{T}f\|^2 \le \int_0^1 \frac{x^3}{3}\left(\int_0^x |f(t)|^2 dt\right) dx \le \frac{1}{3}\int_0^1 \left(\int_0^1 |f(t)|^2 dt\right) dx \le \frac{1}{3} int_0^1 |f(t)|^2 = \frac{1}{3}\|f\|^2,$$

therefore,

$$\|\mathcal{T}\| \le \frac{1}{\sqrt{3}} \sup_{\|f\|\le 1} \|f\| = \frac{1}{\sqrt{3}}.$$

iii) The equation can be written $f = g + \mathcal{T}f$ which equivalent to

$$(I - \mathcal{T})f = g.$$

Since $\|\mathcal{T}\| < 1$ then $I - \mathcal{T}$ is invertible and the equation has a unique solution

$$f = (I - \mathcal{T})^{-1} g.$$

iv) Since $I - 1\mathcal{T}$ is invertible, then $1 \in \rho(\mathcal{T})$.