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Differential Geometry
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Examination time: 1h

EXERCISE 1:

Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (x - \cos y, x + \sin y).$$

(1) Compute the Jacobian matrix of f at $p = (x_0, y_0) \in \mathbb{R}^2$.
(2) Is f an immersion at the point $(0, \pi)$? Is it submersion at the point $(0, \pi/4)$?
(3) Determine the subset $U \subset \mathbb{R}^2$ such that f is a local diffeomorphism at any point of U.
(4) Show that U is an open set.
(5) Show that f is not injective.
(6) Is f a global diffeomorphism?

EXERCISE 2:

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let $G \subset \mathbb{R}^3$ be its graph, i.e. G is given by

$$G = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = g(x, y)\}.$$

(1) Give a function f such that $G = f^{-1}(0)$ (precise its domain and range).
(2) Compute $J_p f$ at a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ and show that it has a maximal rank.
(3) Show that G is a smooth submanifold of $\mathbb{R}^3$. What is its dimension?
(4) Determine the tangent space to G at p.

EXERCISE 3:

Consider the submanifold $\mathscr{E} \subset \mathbb{R}^3$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, c are positive real number. The submanifold $\mathscr{E}$ is called the ellipsoid.

(1) When $a = b = c = 1$, what is $\mathscr{E}$?
(2) what is the dimension of $\mathscr{E}$?
(3) Determine the tangent space to $\mathscr{E}$ at $p = (x_0, y_0, z_0) \in \mathscr{E}$.
(4) Show that the map $X : \mathscr{E} \to \mathbb{R}^3$ given by

$$X(x, y, z) = (\frac{ay}{b}, -\frac{bx}{a}, 0)$$

induces a vector field on $\mathscr{E}$.

(5) Show that the curve $\gamma =]-1, 1[\to \mathscr{E}$ given by

$$\gamma(t) = (a \sin(t), b \cos(t), 0)$$

is an integral curve for X.

Good Luck!
ZELACI HACEN

—————— Solution ——————

Exercise 1:

Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (x - \cos y, x + \sin y).$$

(1) The Jacobian matrix of f at $p = (x_0, y_0) \in \mathbb{R}^2$ is

$$J_p f = \begin{pmatrix} 1 & \sin y_0 \\ 1 & \cos y_0 \end{pmatrix}$$

(2) For the point $(0, \pi)$, we have $J_p f = \begin{pmatrix} 1 & \sin \pi = 0 \\ 1 & \cos \pi = -1 \end{pmatrix}$ whose determinant is $-1 \neq 0$, so $d_p f$ has a maximal rank, hence it is injective. So f is an immersion at this point. For $p = (0, \pi/4)$, we have $J_p f = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ which has a rank equals 1, so $d_p f$ is not surjective. So f is not submersion.

(3) The determinant of $J_p f$ is $\cos y_0 - \sin y_0$. So f is a local diffeomorphism at a point (x_0, y_0) iff $\cos y_0 \neq \sin y_0$, which is equivalent to $y_0 \neq \pi/4 + n\pi$, for any $n \in \mathbb{Z}$. So $U = \mathbb{R}^2 \setminus \{(x, \pi/4 + n\pi) | x \in \mathbb{R}, n \in \mathbb{Z}\}$.

(4) U is an open set because it is the union of open intervals.

(5) We have $f(0, 0) = f(0, 2\pi)$. So f is not injective.

(6) Since f is not injective, it is not a global diffeomorphism.

Exercise 2:

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let $G \subset \mathbb{R}^3$ be its graph, i.e. G is given by

$$G = \{(x, y, z) \in \mathbb{R}^3 \,|z = g(x, y)\}.$$

(1) The function $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(x, y, z) = g(x, y) - z$.

(2) The matrix $J_p f$ at a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ is given by

$$(\frac{\partial g}{\partial x}(x_0, y_0), \frac{\partial g}{\partial y}(x_0, y_0), -1)$$

which is clearly has a maximal rank.

(3) Since $G = f^{-1}(0)$ and since $d_p f$ has a maximal rank for any p, 0 is ragular point. So G is a smooth submanifold of $\mathbb{R}^3$. Its dimension is 2.

(4) The tangent space to G at p is given by

$$T_p G = Ker(d_p f) = \{(x, y, z)| \frac{\partial g}{\partial x}(x_0, y_0)x + \frac{\partial g}{\partial y}(x_0, y_0)y - z = 0\}.$$

Exercise 3:

Consider the submanifold $\mathscr{E} \subset \mathbb{R}^3$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, c are positive real number. The submanifold $\mathscr{E}$ is called the ellipsoid.

(1) When $a = b = c = 1$, The submanifold $\mathscr{E}$ is the sphere S^2.

(2) The dimension of $\mathscr{E}$ is 2.

(3) Tangent space to $\mathscr{E}$ at $p = (x_0, y_0, z_0) \in \mathscr{E}$ is given by

$$T_p\mathscr{E} = \{(x, y, z) | \frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z + 0\}$$

(4) We have $X(p) = X(x_0, y_0, z_0) = (\frac{ay_0}{b}, \frac{bx_0}{a}, 0) \in T_p\mathscr{E}$ (just replace on the equation above). So X induces a vector field on $\mathscr{E}$.

(5) The curve γ verifies

$$X(\gamma) = (a \cos t, -b \sin t, 0)$$

and

$$\gamma'(t) = (a \cos t, -b \sin t, 0).$$

So $X(\gamma) = \gamma'$. Hence γ is an integral curve for X.