University Hama Lakhdar, El Oued. | Differential Geometry
Mathematics department. | 09/01/2022
Master 1 Final Exam | 1 hour

Last name: .... First name: .... Group: .....

EXERCISE 1: (7pt)
Answer with true or false. If it is false, correct it.

(1) A continuous bijective map is called a homeomorphism.

(2) If (φ, O) is a chart on a topological space M then φ is bijective.

(3) Any atlas $\mathscr{A}$ on M is contained in a maximal one.

(4) The disjoint union $\bigsqcup_{p\in M} T_p^*M$ is a vector bundles of rank $2\dim M$.

(5) The projective space $\mathbb{P}^2$ is orientable while the Möbius strip is not.

(6) If E and F are vector bundles over M then $E \cup F$ is again a vector bundle.

(7) Every smooth manifold can be given a Riemannian metric.

EXERCISE 2: (13pt)
Consider the cylinder $\mathcal{C} = \{(x; y; z) \in \mathbb{R} \,|x^2 + y^2 = 1\} \subset \mathbb{R}^3$ with the induced topology, and let $\varphi_1 : \mathbb{R}^2 \to \mathcal{C}$ and $\varphi_2 : \mathbb{R}^2 \to \mathcal{C}$ given by

$$\varphi_1(u, v) = (\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, v).$$

$$\varphi_2(u, v) = (\frac{u^2-1}{1+u^2}, \frac{-2u}{1+u^2}, v).$$

Denote by V_i the image of φ_i (for $i = 1, 2$).

(1) Show that φ_1 is injective.
(2) Is φ_1 surjective? Justify your answer.
(3) Describe precisely V_1 the image of φ_1, and deduce that it is an open in $\mathcal{C}$.
(4) Deduce that $\varphi_1 : \mathbb{R}^2 \to V_1$ is bijective.
(5) Show that the map $\varphi_1^{-1} : V_1 \to \mathbb{R}^2$ given by

$$(x, y, z) \mapsto (\frac{y}{1+x}, z)$$

is the inverse map of φ_1.

$$\left(\begin{array}{l}\text{Use the fact that } x^2+y^2=1 \\ \text{which induces } \dfrac{y^2}{(1+x)^2} = \dfrac{1-x}{1+x}\end{array}\right)$$

(6) Show that $\varphi_1 : \mathbb{R}^2 \to V_1$ is a homeomorphism and deduce that (φ_1^{-1}, V_1) is a chart on $\mathcal{C}$.
(7) Prove that the collection $\mathcal{A} = \{(\varphi_1^{-1}, V_1), (\varphi_2^{-1}, V_2)\}$ is an atlas on $\mathcal{C}$ (we admit that (φ_2^{-1}, V_2) is a chart).
(8) Deduce that $\mathcal{C}$ is an abstract manifold. What is its dimension?

Good luck!

University Hama Lakhdar, El Oued.
Mathematics department.

Differential Geometry
11/03/2021

Solution

EXERCISE 1: (7pt)
Answer with true or false. If it is false, correct it.

(1) A continuous bijective map is called a homeomorphism. False: A continuous bijective map whose inverse is continuous is called a homeomorphism.

(2) If (φ, O) is a chart on a topological space M then φ is bijective. True

(3) Any atlas $\mathscr{A}$ on M is contained in a maximal one. True

(4) The disjoint union $\bigsqcup_{p\in M} T_p^*M$ is a vector bundles of rank $2\dim M$. False: of rank $\dim M$.

(5) The projective space $\mathbb{P}^2$ is orientable while the Möbius strip is not. False: Both $\mathbb{P}^2$ and the Möbius strip are not orientable.

(6) If E and F are vector bundles over M then $E \cup F$ is again a vector bundle. False: $E \cup F$ is not a vector bundle in general.

(7) Every smooth manifold can be given a Riemannian metric. True

EXERCISE 2: (13pt)
Consider the cylinder $\mathcal{C} = \{(x; y; z) \in \mathbb{R} \,|x^2 + y^2 = 1\} \subset \mathbb{R}^3$ with the induced topology, and let $\varphi_1 : \mathbb{R}^2 \to \mathcal{C}$ and $\varphi_2 : \mathbb{R}^2 \to \mathcal{C}$ given by

$$\varphi_1(u, v) = (\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, v).$$

$$\varphi_2(u, v) = (\frac{u^2-1}{1+u^2}, \frac{-2u}{1+u^2}, v).$$

Denote by V_i the image of φ_i (for $i = 1, 2$).

(1) Show that φ_1 is injective. let $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ s.t. $\varphi_1(u_1, v_1) = \varphi_1(u_2, v_2)$, then $\frac{1-u_1^2}{1+u_1^2} = \frac{1-u_2^2}{1+u_2^2}$, $\frac{2u_1}{1+u_1^2} = \frac{2u_2}{1+u_2^2}$ and $v_1 = v_2$. The first one implies $u_1^2 = u_2^2$, replacing this in the second one we get $u_1 = u_2$.

(2) Is φ_1 surjective? Justify your answer. No, because $(-1, 0, v)$ doesn't belong to its image, for any v.

(3) Describe precisely V_1 the image of φ_1, and deduce that it is an open in $\mathcal{C}$. $V_1 = \{(x, y, z)|x \neq -1, x^2 + y^2 = 1\}$. So geometrically, it is the cylinder deprived of the line with equations $x = -1$, $y = 0$. Since this line is closed, V_1 is open.

(4) Deduce that $\varphi_1 : \mathbb{R}^2 \to V_1$ is bijective. By equation (1), φ_1 is injective, and by definition of V_1, φ_1 is surjective on V_1. So it is bijective.

(5) Show that the map $\varphi_1^{-1} : V_1 \to \mathbb{R}^2$ given by

$$(x, y, z) \mapsto (\frac{y}{1+x}, z)$$

is the inverse map of φ_1. We have for all $(u,v) \in \mathbb{R}^2$ and all $(x,y,z) \in V_1$:

$$\begin{aligned}\varphi_1^{-1} \circ \varphi_1(u,v) &= \varphi_1^{-1}(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, v) \\ &= (\frac{\frac{2u}{1+u^2}}{1+\frac{1-u^2}{1+u^2}}, v) = (u,v).\end{aligned}$$

and

$$\begin{aligned}\varphi_1 \circ \varphi_1^{-1}(x,y,z) &= \varphi_1(\frac{y}{1+x}, z) \\ &= (\frac{1-(\frac{y}{1+x})^2}{1+(\frac{y}{1+x})^2}, \frac{2(\frac{y}{1+x})}{1+(\frac{y}{1+x})^2}, z) \\ &= (\frac{1-(\frac{1-}{1+x})}{1+(\frac{1-x}{1+x})}, \frac{2(\frac{y}{1+x})}{1+(\frac{1-x}{1+x})}, z) \\ &= (x,y,z).\end{aligned}$$

(6) Show that $\varphi_1 : \mathbb{R}^2 \to V_1$ is a homeomorphism and deduce that (φ_1^{-1}, V_1) is a chart on $\mathcal{C}$. By the previous questions, φ_1 is bijective. Moreover, from the expressions of φ_1 and φ_1^{-1}, we see that both are continuous, since they are given by well defined rational polynomials. Since $\varphi_1^{-1} : V_1 \to \mathbb{R}^2$ and both V_1 and $\mathbb{R}^2$ are open, we deduce that (φ_1^{-1}, V_1) is a chart on $\mathcal{C}$.

(7) Prove that the collection $\mathcal{A} = \{(\varphi_1^{-1}, V_1), (\varphi_2^{-1}, V_2)\}$ is an atlas on $\mathcal{C}$. One can see that V_2 is the cylinder $\mathcal{C}$ without the line whose equations are $x = 1$ and $y = 0$. So clearly $V_1 \cap V_2 = \mathcal{C}$. Now the composition $\varphi_1^{-1} \circ \varphi_2$ is given on $\varphi_2^{-1}(V_1 \cap V_2) = \{(u,v) | u \neq 0\} \subset \mathbb{R}^2$ by

$$(u,v) \mapsto (\frac{\frac{-2u}{1+u^2}}{1+\frac{u^2-1}{1+u^2}}, v) = (\frac{-1}{u}, v),$$

which is clearly smooth.

(8) Deduce that $\mathcal{C}$ is an abstract manifold. What is its dimension? We have an atlas $\mathcal{A}$ on $\mathcal{C}$, so we have a maximal atlas $\mathcal{A}$, and $\mathcal{C}$ is clearly Hausdorff, so $(\mathcal{C}, \mathcal{A})$ is an abstract manifold. Its is dimension is 2.