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Differential Geometry
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Solution

EXERCISE 1:

Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (e^x + y, e^y - x).$$

(1) Compute the Jacobian matrix of f at $p = (x_0, y_0) \in \mathbb{R}^2$.

$$J_p(f) = \begin{pmatrix} e^{x_0} & 1 \\ -1 & e^{y_0} \end{pmatrix}$$

(2) Show that f is a local diffeomorphism at any point of $\mathbb{R}^2$. We have $\det(J_p(f)) = e^{x_0+y_0} + 1 > 0$, so by the inverse function theorem, we deduce that f is a local diffeomorphism at any $p \in \mathbb{R}^2$.

(3) Show that f is injective. let $(x,y), (x',y') \in \mathbb{R}^2$ s.t $f(x,y) = f(x',y')$. Then $e^x - e^{x'} = y' - y$ and $e^y - e^{y'} = x - x'$. Assume that $x > x'$, so $e^y > e^{y'}$, since the function $x \mapsto e^x$ is strictly increasing we deduce that $y > y'$, so $e^{x'} > e^x$ but this implies that $x' > x$, a contradiction. Similarly $x' < x$ implies a contradiction. So $x = x'$ and $y = y'$.

(4) Show that f is surjective. if $f(x,y) = (a,b)$, then $e^{e^y - b} + y = a$, the function $g : y \mapsto e^{e^y - b} + y$ is a continuous strictly increasing ($g' > 0$) such that $\lim_{\pm\infty} g = \pm\infty$. So it is surjective from $\mathbb{R} \to \mathbb{R}$. Hence the equation $f(x,y) = (a,b)$ has a solution.

(5) Deduce that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a global diffeomorphism. Since f is a bijective local diffeomorphism, then it is a global diffeomorphism.

EXERCISE 2:

Let $M \subset \mathbb{R}^3$ be the subset given by

$$M = \{(x,y,z) \in \mathbb{R}^3 \,|x \sin(z) - y\cos(z) = 0\}.$$

(1) Give a function f such that $M = f^{-1}(0)$. $f(x,y,z) = x \sin z - y \cos z$. $f : \mathbb{R}^3 \to \mathbb{R}$.

(2) Compute $J_p(f)$ at a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ and show that it has a maximal rank. We have $J_p(f) = (\sin(z), -\cos(z), x\cos(z) + y\sin(z))$, since $\sin(z)$ and $\cos(z)$ do not vanish simultaneously, we deduce that $rk(J_p(f)) = 1$ which is maximal.

(3) Show that M is a submanifold of $\mathbb{R}^3$ and give its dimension. By question 2), we deduce that 0 is a regular value for f, hence $f^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^3$ of dimension 2.

(4) Determine precisely the tangent space T_pM, where $p = (x_0, y_0, z_0) \in M$. We have $T_pM = \mathrm{Ker}(d_pf)$, since $d_pf(x,y,z) = x\sin(z_0) - y\cos(z_0) + z(x_0\cos(z_0) + y_0\sin(z_0))$, we get

$$T_pM = \{(x,y,z) \in \mathbb{R}^3 \,|\, x\sin(z_0) - y\cos(z_0) + z(x_0\cos(z_0) + y_0\sin(z_0)) = 0\}.$$

EXERCISE 3:

Let $M = \mathbb{R}^2$.

(1) Determine $\Omega^1(M)$ and $\Omega^2(M)$. $\Omega^1(M) = C^\infty(M)dx + C^\infty(M)dy$, and $\Omega^2(M) = C^\infty(M)dx \wedge dy$.

(2) Let $\omega = (x + y^2)dx + (y + x^2)dy$. Calculate $d\omega$. Is ω a closed 1−form? $d\omega = \frac{\partial(x+y^2)}{\partial y} dy \wedge dx + \frac{\partial(y+x^2)}{\partial x} dx \wedge dy = 2(x-y)dx \wedge dy$. In particular ω is not closed.

(3) Let $\omega' = x^2y^3dx + y^2x^3dy$. Is ω' exact 1−form? If yes, determine $f \in C^\infty(M)$ such that $df = \omega'$. we have $\frac{\partial(x^2y^3)}{\partial y} = 3x^2y^2 = \frac{\partial(x^3y^2)}{\partial x}$, so ω' is an exact 1-form. So we look for a function f such that $\frac{\partial f}{\partial x} = x^2y^3$ and $\frac{\partial f}{\partial y} = x^3y^2$. One can take $f(x, y) = x^3y^3/3$. So $df = \omega'$.

EXERCISE 4:

Let V be an $\mathbb{R}$−vector space of dimension d, and let $k \in \{1, \cdots, d\}$.

(1) Assume that $(e_1, \cdots, e_d)$ is a basis for V. Determine a basis for $\Lambda^k V$. A basis is given by $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \,|\, i_j \in \{1, \cdots, d\}$ such that $i_1 < i_2 < \cdots < i_k\}$.

(2) Deduce the dimension of $\Lambda^k(V)$. $\dim \Lambda^k(V) = \binom{k}{d}$.

(3) Show that the map $\Psi : \Lambda^{d-1}V \to V^*$ given by

$$\Psi(\alpha) : V \to \mathbb{R}$$
$$x \mapsto x \wedge \alpha$$

is an isomorphism. We have for any $\lambda \in \mathbb{R}$, $\alpha, \beta \in \Lambda^{d-1}V$ and any $x \in V$

$$\begin{aligned} \Psi(\alpha + \lambda\beta)(x) &= x \wedge (\alpha + \lambda\beta) \\ &= x \wedge \alpha + \lambda x \wedge \beta \\ &= \Psi(\alpha)(x) + \lambda\Psi(\beta), \end{aligned}$$

hence Ψ is linear. Moreover, if $\Psi(\alpha)(x) = 0$ for any x, then, by taking $x \in \{e_1, \cdots, e_d\}$, we deduce that $\alpha = 0$ so Ψ is injective. Since $\dim \Lambda^{d-1}V = \dim V^*$ we deduce that Ψ is also surjective, hence it is an isomorphism.