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El Oued, le 29-05-2021
1ère Master académique
Durée : soixante minute

Correction du contrôle du 2ème semestre de modélisation

Exercice1: (06 points)

1) $X_m X_m = X_1X_1 + X_2X_2 + X_3X_3 = 14$

2) $A_{mn}X_n = AX = X = \begin{pmatrix} 7 \\ 8 \\ 12 \end{pmatrix}$

3) $A_{mn}A_{mn} = \text{trace}(AA^T) = 28$

4) $\delta_{kl}A_{lk} = A_{kk} = \text{trace}(A) = 5$

5) $\varepsilon_{ijk}A_{jk} = \varepsilon_{i1k}A_{1k} + \varepsilon_{i2k}A_{2k} + \varepsilon_{i3k}A_{3k} = (\varepsilon_{i12}A_{12} + \varepsilon_{i13}A_{13}) + (\varepsilon_{i21}A_{21} + \varepsilon_{i23}A_{23}) + (\varepsilon_{i31}A_{31} + \varepsilon_{i32}A_{32}) = V_i$

$$\left.\begin{array}{l} \text{Pour } i=1, \text{ on a }, V_1 = \varepsilon_{123}A_{23} + \varepsilon_{132}A_{32} = A_{23} - A_{32} = 2 \\ \text{Pour } i=2, \text{ on a }, V_2 = \varepsilon_{213}A_{13} + \varepsilon_{231}A_{31} = -A_{13} + A_{31} = 1 \\ \text{Pour } i=3, \text{ on a }, V_3 = \varepsilon_{312}A_{12} + \varepsilon_{321}A_{21} = A_{12} - A_{21} = 0 \end{array}\right\} \Rightarrow \varepsilon_{ijk}A_{jk} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Exercice 2: (06 points)

1) $\vec{T}(M,\vec{n}) = \sigma n = \begin{pmatrix} 0 & 12 & 12 \\ 12 & 0 & 12 \\ 12 & 12 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} = \begin{pmatrix} 8\sqrt{3} \\ 8\sqrt{3} \\ 8\sqrt{3} \end{pmatrix} \Rightarrow \vec{T}(M,\vec{n}) = 8\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)$

2) $\vec{T}(M,\vec{n}) = \sigma_n\vec{n} + \vec{\tau} \Rightarrow \sigma_n = \vec{T}(M,\vec{n}).\vec{n} = (8\sqrt{3})\frac{\sqrt{3}}{3} + (8\sqrt{3})\frac{\sqrt{3}}{3} + (8\sqrt{3})\frac{\sqrt{3}}{3} = 24$

$\left\|\vec{T}(M,\vec{n})\right\|^2 = \sigma_n^2 + \|\vec{\tau}\|^2 \Rightarrow \|\vec{\tau}\| = \sqrt{\left\|\vec{T}(M,\vec{n})\right\|^2 - \sigma_n^2}$

$\Rightarrow \|\vec{\tau}\| = \sqrt{(8\sqrt{3})^2 + (8\sqrt{3})^2 + (8\sqrt{3})^2 - 24^2} = \sqrt{24^2 - 24^2} = 0 \Rightarrow \vec{\tau} = \vec{0} \Rightarrow \vec{T}(M,\vec{n}) = \sigma_n\vec{n}$

3. On a $\vec{T}(M,\vec{n}) = \sigma_n\vec{n} \Rightarrow \sigma\vec{n} = \sigma_n\vec{n}$ donc σ_n et $\vec{n}$ sont la contrainte et la direction principales

Exercice 3: (08 points)

1. On a $\vec{T}(M,\vec{n}) = \sigma\vec{n}$ où $\vec{n} = \frac{\vec{u}}{\|\vec{u}\|}$, $\vec{u} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$, $(a,b,c) \neq (0,0,0)$

$$\vec{T}(M,\vec{n}) = \sigma\vec{n} = 0_{\mathbb{R}^3} \Rightarrow \sigma\vec{u} = 0_{\mathbb{R}^3} \Rightarrow \begin{pmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \sigma_{11}a + 2b + c = 0 \\ 2a + 2c = 0 \\ a + 2b = 0 \end{cases} \Rightarrow \begin{cases} \sigma_{11}a - a - a = 0 \\ c = -a\ (a \neq 0) \\ 2b = -a \end{cases} \Rightarrow \sigma_{11} = 2$$

$\vec{n} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3)}{\sqrt{a^2 + b^2 + c^2}}$, on a pour $a \neq 0$, $c = -a$ et $b = -\frac{a}{2}$ donc

$$\vec{n} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{a\vec{e}_1 - \frac{a}{2}\vec{e}_2 - a\vec{e}_3}{\sqrt{a^2 + \frac{a^2}{4} + a^2}} = \frac{a(\vec{e}_1 - \frac{1}{2}\vec{e}_2 - \vec{e}_3)}{\sqrt{\frac{9a^2}{4}}} = \frac{2}{3}(\vec{e}_1 - \frac{1}{2}\vec{e}_2 - \vec{e}_3)$$

2.

$$\begin{vmatrix} 2-\sigma & 2 & 1 \\ 2 & -\sigma & 2 \\ 1 & 2 & -\sigma \end{vmatrix} = (2-\sigma)(\sigma^2 - 4) - 2(-2\sigma - 2) + 4 + \sigma = 0 \Leftrightarrow -\sigma(\sigma^2 + 2\sigma - 9) = 0 \Leftrightarrow \begin{cases} \sigma_1 = 1 + \sqrt{10} \\ \sigma_2 = 0 \\ \sigma_3 = 1 - \sqrt{10} \end{cases}$$

3. la contrainte tangentielle maximale $\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1 + \sqrt{10} - (1 - \sqrt{10})}{2} = \sqrt{10}$