التصحيح النموذجي لامتحان 2021

1

(6 نقاط)

ت 1: أثبت أن الدالة $f(z) = sh(z)$ دالة تحليلية.

الدالة تحليلية على $\mathbb{C}$ $\Leftarrow$ قابلة للاشتقاق على $\mathbb{C}$. (0,5)

قابلة للاشتقاق عند أي نقطة $\Leftarrow$ تحقق شرطي كوشي-ريمان. (0,5)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad (1)$$

$$f(z) = u + iv = sh(z) = \frac{e^{z} - e^{-z}}{2} = \frac{1}{2}\left(e^{x+iy} - e^{-x-iy}\right) \quad (0,5)$$

$$= \frac{1}{2}\left(e^{x}e^{iy} - e^{-x}e^{-iy}\right) = \frac{1}{2}\left(e^{x}(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)\right) \quad (0,5)$$

$$= \frac{1}{2}\left(\cos y\left(e^{x} - e^{-x}\right) + i\sin y\left(e^{x} + e^{-x}\right)\right) = \underbrace{\cos y\, sh x}_{} + i\underbrace{\sin y\, ch x}_{}. \quad (0,5)$$

$$f(z) = u(x,y) + i\,v(x,y).$$

$\dfrac{\partial u}{\partial x} = \cos y\, ch x$ (0,5) ، $\dfrac{\partial v}{\partial y} = \cos y\, ch x$ (0,5)

$\dfrac{\partial u}{\partial y} = -\sin x\, sh x$ ، $\dfrac{\partial v}{\partial x} = \sin y\, sh x$.

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$ محققين (1)

شرطي كوشي ريمان محققين عند كل نقاط الفضاء $\mathbb{C}$ $\Leftarrow$

$f(z) = sh(z)$ دالة تحليلية على $\mathbb{C}$.

ت 02 : باستعمال نظرية كوشي :

$$\oint_{(C)} \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0) \Leftarrow$$ إذا كانت $f(z)$ تحليلية داخل (C)

و z_0 داخل المسار (C)

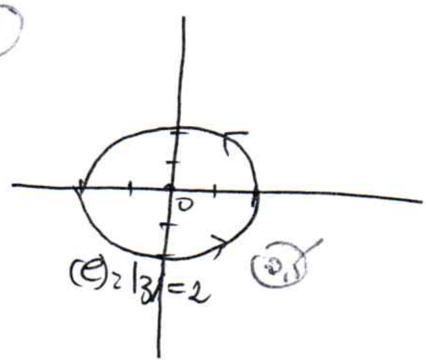
$$\oint_{(C)} \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!}\, f^{(n)}(z_0) \Leftarrow$$ 0,5

1)
$$\oint_{(C):|z|=2} \frac{e^{2z}}{(z+i)^2}\,dz =$$

$f(z)=e^{2z}$ الدالة 0,5

النقطة z_0 : $z_0=-i$

داخل المسار 0,5



$(C): |z|=2$ 0,5

$$\oint_{(C)} \frac{e^{2z}}{(z+i)^2}\,dz = \frac{2\pi i}{1!}\, f^{(1)}(z_0) \quad 0,5$$

$$= 2\pi i\cdot\left(2e^{2z}\right)\Big|_{z=-i} = 2\pi i\left(2e^{-2i}\right) = 4\pi i\left(\cos 2 - i\sin(2)\right) \quad 0,5$$

$$\boxed{\oint_{C:|z|=2} \frac{e^{2z}}{(z+i)^2} = 4\pi\left(\sin(2) + i\cos(2)\right)} \quad 1$$

2)
$$\oint_{(C):|z-1|=1} \frac{dz}{z^3-1} = 2\pi i\cdot f(z_0) \quad 0,5$$

$z^3-1=0 \Rightarrow z^3=1=e^{i2k\pi}$

$\Rightarrow z_k = e^{i\frac{2k\pi}{3}} \Rightarrow \begin{cases} z_1 = 1 \\ z_2 = e^{i\frac{2\pi}{3}} = -\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ z_3 = e^{i\frac{4\pi}{3}} = -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{cases}$ 1

$$= \oint_{(C)} \frac{dz}{(z-z_1)(z-z_2)(z-z_3)}$$

$z_1 = 1$ داخل المسار

z_2 و z_3 خارج المسار $\Leftarrow$ 0,5

$(C): |z-1|=1$

$f(z) = \frac{1}{(z-z_2)(z-z_3)}$ تحليلية داخل (C) لأن z_2 و z_3 خارج 0,5

$$\Rightarrow \oint \frac{dz}{z^3-1} = 2\pi i\, f(z_1) = 2\pi i \times \frac{1}{(z_1-z_2)(z_1-z_3)} = 2\pi i\cdot\frac{1}{\left(1+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} = \frac{2\pi i}{\left(\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$\oint_{(C):|z-1|=1} \frac{dz}{(z^3-1)} = 2\pi i \times \frac{1}{\frac{9}{4}+\frac{3}{2}} = \frac{(2\pi i)\,4}{(9+6)} = \frac{8\pi i}{15}$$ (1)

ت 03 : (6 نقاط) باستعمال نظرية الرواسب :

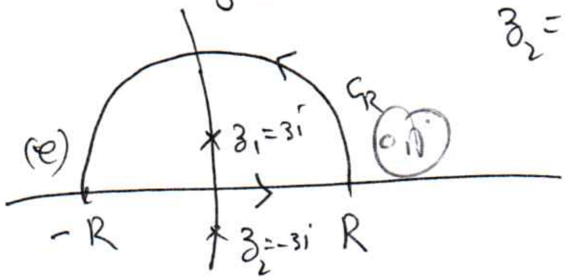
$$\oint_{(C)} f(z)\,dz = 2\pi i \sum_{z_i} \mathrm{Res}(f, z_i)$$ (0,5)

$$\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+9)^2}\,dx. \qquad f(z) = \frac{z^2}{(z^2+9)^2}.$$ (0,5)

* دالة مركبة تحليلية ماعدا عند اقطابها.

$(z^2+9)^2 = 0$

$z^2 = -9 \Rightarrow z_1 = 3i,\ z_2 = -3i$ (0,5) اقطاب من الرتبة $m=2$



* المسار

$z_1 = 3i$ داخل (C).

$$\oint_{(C)} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, z_1)$$ (0,5) *

$$\int_{-R}^{R} \frac{z^2}{(z^2+9)^2}\,dz + \int_{C_R} \frac{z^2}{(z^2+9)^2}\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+9)^2}\,dx + \int_0^{\pi} \frac{R^2 e^{2i\theta}}{(R^2e^{2i\theta}+9)^2}\,Rie^{i\theta}\,d\theta$$

$z = x$ (0,5) $z = Re^{i\theta}$ (0,5) $\to 0$

$R \longrightarrow \infty$ *

$$\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+9)^2}\,dx = 2\pi i\,\mathrm{Res}(f, z_1)$$ (0,5)

$$\mathrm{Res}(f, z_1) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}\left[(z-z_1)^m f(z)\right]_{z=z_1} = \frac{\partial}{\partial z}\left[(z-z_1)^2 f(z)\right]_{z=z_1} = \frac{\partial}{\partial z}\left[\frac{z^2}{(z-z_2)^2}\right]_{z=z_1}$$ (0,5)

$$= \frac{2z(z-z_2)^{\cancel{2}} - 2z^2\cancel{(z-z_2)}}{(z-z_2)^4}\Bigg|_{z=z_1} = \frac{2z_1(z_1-z_2) - 2z_1^2}{(z_1-z_2)^3} = \frac{2(3i)(2(3i)) - 2(3i)^2}{2(3i)^3} = \frac{\cancel{2}(3i)^{\cancel{2}}}{\cancel{2}(3i)^{\cancel{3}}}$$

$$\boxed{\mathrm{Res}(f, z_1) = \left(\frac{1}{3i}\right)}$$ (0,5)

$$\boxed{\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+9)^2}\,dx = 2\pi \cancel{i} \times \frac{1}{3\cancel{i}} = \frac{2\pi}{3}}$$ (1)