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Ministry of Higher Education and Scientific Research
Echahid Hamma Lakhdar University – El Oued

Laboratory of Operator Theory and PDE: Foundations and Applications

Organize

3rd International conference in operator theory, PDE and Applications

March 12 - 13, 2019

Themes

- *Operator theory,*
- *PDE and applications,*
- *Numerical analysis.*

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Deadline of submission: Jan 15, 2019, *contacts: farehabdelf@gmail.com, aissaouiadel@gmail.com.*

Echahid Hamma Lakhdar University – El Oued
Faculty of Exact Sciences. Mathematics Department
Laboratory of Operator Theory and PDE: Foundations and Applications

A3M' 2021

Program of the workshop

Tuesday March 30, 2021

I- Opening ceremony 08h30 – 09h00

II- Plenary lectures

1	9:15 -10:00	Dr. Hacen Zelaci	Strange duality at level one for the anti invariant vector bundles
	10:00-10:25	Tea Break	
2	10:30-11:15	Pr. Salim Messaoudi	On the general decay for a Class of Viscoelastic Weakly Dissipative Second-Order Systems
3	11:20-12:00	Pr. Nadjib Boussetila	Approximation numérique de certaines équations intégrales de première espèce par des méthodes spectrales

III- Lunch break

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Workshop: Numerical analysis.

Workshop Chair: Beggas Mohammed

First day Tuesday 30 March 2021

N			
1	Ismail Merabet	Stabilized finite element method for non-homogeneous Timoshenko beam	13:00-13:45
2	El Amir Djeffal	Complexity analysis of a convex quadratic semidefinite optimization problem using a projective transformation	13:50-14:05
3	Hadda Benaddi	The method (HPM) for solving some problems of heat-like equations with non local boundary conditions.	14:10-14:25
4	Nidal Dib	Numerical simulations of the in viscid stochastic burgers equation on a bounded domains	14:30-14:45
5	Bouabsa Wahiba	Mean square error of the local linear estimation of the conditional mode function for functional data	14:50-15:05
6	Houdeifa Melki	La résolution du problème de Cauchy elliptique mal posé.	15:10-15:25
7	Rezzag Bara Rihana	A mixed formulation of flexural prestressed shell model	15:30-15:45
8	Wassila Ghecham	Feedback boundary stabilization of the Schrodinger equation with interior delay	15:50-16:05
Tea Break			
9	Amel Redjil	The Stochastic Control Theory in the G-frame work	16:30-16:45
10	Walid Remili	Numerical solution of high-order ordinary differential equations on the half-line	16:50-17:05
11	Dalila Takouk and Rebiha Zeghdane	Some numerical approaches based on compactly supported RBF and FDM method for solving PDEs.	17:10-17:25
12	Hakima Miloudi	Stochastic maximum principle for partially observed optimal control problems of general McKean-Vlasov differential equations	17:30-17:45

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Workshop: Mathematical Modeling.

Workshop Chair: Hadj Ammar Tidjani

First day Tuesday 30 March 2021

1	Aziza Bachmar	Variational Analysis of a Dynamic Electro viscoelastic Problem with Friction	13:00-13:45
2	Ilyas Boukaroura	A variational study for a contact problem between two thermo-viscoelastic bodies	13:50-14:05
3	Ahmed Hamidati	Dynamic contact problem with wear for an elastic-viscopalastic body and damage	14:10-14:25
4	Abdelkader Saadallah	Asymptotic convergence of a generalized non-newtonian fluid with tresca boundary conditions	14:30-14:45
5	Abdelmoumene Djabi	Etude variationnelle d'un problème de contact avec usure et endommagement	14:50-15:05
6	Nadhir Chougui	Analysis of quasistatic viscoelastic viscoplastic piezoelectric contact problem with friction and adhesion	15:10-15:25
7	Safa Gheriani	Analysis of a unilateral contact problem	15:30-15:45
8	Tikialine Belgacem	Exponential decay for a nonlinear axially moving viscoelastic string	15:50-16:05
9	Rouibah Khaoula	Iterative continuous collocation method for solving nonlinea r volterra integro- differential equations	16:10-16:25
10	Islah Atmania	L'existence et l'unicité de problème d'évolution non classique	16:30-16:45
11	Amar Bougoutaia	Some ideals of linear and non-linear operators and their applications	16:50-17:05
12	Nadjate Djellab	Approximate method for oxygen diffusion and absorption in sick cell	17:10-17:25

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Laboratory of Operator Theory and PDE: Foundations and Applications

Workshop: Operators theory

Workshop Chair: Guedda Lamine

First day Tuesday 30 March 2021

1	Daifia Ala Eddine	Galerkin method for the higher dimension Boussinesq equation non linear with integral condition	13:00-13:15
2	Asma Hammou	Extension and lifting polynomials on banach spaces	13:20-13:35
3	Amar Bougoutaia	Some ideals of linear and non-linear operators and their applications	13:40-13:55
4	Khedidja Kherchouche	Iterative Collocation Method for Solving a class of Nonlinear Weakly Singular Volterra Integral Equations	14:00-14:15
5	Djamel Abid	Multiple solutions for the p-fracional laplacian with critical growth	14:20-14:35
6	Hafida Laib	Approximate solution of systems of nonlinear volterra delay integro-differential equations	14:40-14:55
7	Faiza Zaamoune	Investigation of bifurcation de Hopf from the method hidden bifurcation in Chua system generated by Transformation	15:00-15:15
8	Hanane Kessal	Quasi-variational inequalities of hyberbolic type for a dynamic viscoelastic contact frictional problem	15:20-15:35
Tea Break			
9	Abdelkrim Bencheikh and Lakhdar Chiter	Operational Matrix Method for fractional Emden-Fowler problem	16:00-16:15
10	Souraya Fareh	Multiplicity results for Steklov problem involving the p(x)-Laplacian	16:20-16:35
11	Bouzid Mansouri	Stability of neutral nonlinear differential equations by fixed point theorem	16:40-16:55
12	Ahmed Zahed	Boundary value problems for Hadamard-caputo Fractional Differential inclusion with nonlocal condition	17:00-17:15
13	Redouane Douaifia	Global existence, asymptotic stability and numerical simulation for a coupled two-cell Schnakenberg reaction-diffusion model	17:20-17:35

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Workshop: PDEs and applications

Workshop Chair: Fareh Abdelfateh

First day Tuesday 30 March 2021

1	Belkasem Said Hoauri	Global well posedness of the Cauchy problem for the Jordan Moore Gibson Thomson equation	13:00-13:25
2	Abdelaziz Soufyane	Stability of one dimensional piezoelectric beams model with magnetic effects	13:30-13:55
3	Djamel Ouchenane	Global well-posedness and exponential stability results of a class of Bresse Timoshenko type systems with distributed delay term	14:00-14:25
4	Oulaia Bouhoufain	Blow up of negative initial-energy solutions for a coupled system of nonlinear hyperbolic equation with variable	14:30-14:45
5	Zineb Khalili	Energy decay for a nonlinear timoshenko-system with infinite history	14:50-15:05
6	Amel Boudiaf	On the control of a nonlinear system of viscoelastic equations	15:10-15:25
7	Imane Boulmerka and Ilhem Hamchi	Existence and properties of solution of the wave equation with general source and damping terms.	15:30-15:45
8	Belhadji Bochra	Blow-up phenomena for a viscoelastic wave equation with Balakrishnan-Taylor damping and logarithmic nonlinearity	15:50-16:05
Tea Break			
9	Ahlem Mesloub	General decay for a viscoelastic Kirchhoff equation with not necessarily decreasing kernel	
10	Fares Yazid	Bresse Timoshenko system: well-posedness, stability and numerical results	16:30-16:45
11	Ahlem Merah and Fatiha Mesloub	La stabilite d'une equation d'onde viscoelastique avec les contions dynamique	
12	Islem Baaziz	General stability result for viscoelastic wave equation with boundary feedback and a nonlinear source	17:10-17:25

Stabilized finite element method for non-homog Timoshenko beam.

I. Merabet[1]

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Abstract

This work deals with the finite element approximation of the Timoshenko system. We prove the existence and the uniqueness of the continuous and the discrete problems. We propose a stabilized finite element method for the corresponding mixed formulation. Numerical tests are included.

Keywords: Finite element approximation, Stabilzation, Timoshenko beams

References

[1] T. Gustafsson, R. Stenberg and J. Videman. Mixed and stabilized finite element method for the obstacle problem. SIAM J. Numer. Anal., 55(6), 2718–2744.(2017).

[2] C. Lovadina,D. Mora and R. Rodriguez. Locking-free finite element method for the buckling problem of a non-homogeneous Timoshenko beam. ESAIM: M2AN Volume 45, Number 4,2011. 603-626.

BRESSE-TIMOSHENKO SYSTEM : WELL-POSSEDNESS AND STABILITY RESULTS

F. YAZID AND D. OUCHENANE

Abstract. In this paper, we consider a Bresse-Timoshenko type system with distributed delay term. Under suitable assumptions, we establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. By using the energy method, we show the exponential stability results for the system with delay in vertical displacement.

1. introduction

In this paper, we deal with a nolinear Bresse-Timoshenko type system with distributed delay, under appropriate assumptions and we study the exponential decay.

It is related to the problem of stability for dissipative models of the Timoshenko type related to the problem of the damage consequences of the so called second spectrum of frequencies, or simply second spectrum. We mention the recent works in [2]-[4] and [10], the distributed delay is not explicitly presented, and therefore it makes sense to consider the problems in this paper. In [18], a Timoshenko system is considered

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu\psi_t = 0 \end{cases} \tag{1}$$

and after an explanation about the meaning of second spectrum based on existing literature, the authors showed that the viscous damping acting on angle rotation of (1).

To the best of our knowledge, the first contribution in that direction was obtained by Manevich and Kolakowski [16]. They analyzed the dynamic of a Timoshenko model where the damping mechanism is viscoelastic. More precisely, they considered the dissipative system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x - \mu_1(\varphi_x + \psi)_{tx} = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) - \mu_2\psi_{ttx} + \mu_1(\varphi_x + \psi)_t = 0 \end{cases} \tag{2}$$

Secondly, based on Elishakoff's papers and collaborators and their studies on truncated versions for classical Timoshenko equations [1] (see also recent contributions of Elishakoff et *al.* [7]-[8]), Almeida Junior and Ramos [2] showed that the total energy for viscous damping acting on angle rotation of the simplified Timoshenko

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Key words and phrases. Bresse-Timoshenko-type systems; Exponential decay; distributed delay term.

system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t = 0 \end{cases} \tag{3}$$

The model is very different from classical Timoshenko system, since it contains three derivatives: two derivatives with respect to time and one derivative with respect to space. The reason behind this is the absence of the second spectrum or non-physical spectrum [1, 8] and its damage consequences for wave propagation speeds [2]. We can find the historical and mathematical observations in [1, 8]. The same results are achieved for a dissipative truncated version where the viscous damping acts on vertical displacement

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{4}$$

Then, in order to get more consistent exponential decay results in light of the absence of second spectrum, Almeida Junior et *al.* [4] considered two cases of dissipative systems for Bresse-Timoshenko type systems with constant delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau) = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{5}$$

For the second one, the authors also proved the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x, t-\tau) = 0 \end{cases} \tag{6}$$

Feng et *al.* [10] considered two cases of dissipative systems for Bresse-Timoshenko type systems with time-varying delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau(t)) = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{7}$$

For the second one, the authors also proved the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x, t-\tau(t)) = 0 \end{cases} \tag{8}$$

A complement to these works, we are working to establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. And prove the exponential decay of two cases of dissipative systems for Bresse-Timoshenko type systems with distributed delay, under appropriate assumptions and we prove these results using the energy method and with the help of convex functions. In the following, let c a positive constant.

2. Distributed delay and viscous damping in vertical displacement

Here, we are concerned with the following system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(p)|\varphi_t\,(x, t-p)\,dp = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{9}$$

where

$$(x,p,t)\in(0,1)\times(\tau_1,\tau_2)\times(0,\infty)$$

Additionally, we consider initial conditions given by

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \varphi_{tt}(x,0)=\varphi_2(x) \\ \varphi_{ttt}(x,0)=\varphi_3(x),\ \psi(x,0)=\psi_0(x), x\in(0,1) \end{cases} \tag{10}$$

where $\varphi_0,\varphi_1,\varphi_2,\psi_0$, are given functions, and boundary conditions of Dirichlet-Dirichlet given by

$$\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\ t>0 \tag{11}$$

Wherever, φ is the transverse displacement of the beam, ψ is the angle of rotation, and $\rho_1,\rho_2,b,\beta>0$ and the integral represents the distributed delay term with $\tau_1,\tau_2>0$ are a time delay, μ_1 is positive constant, μ_2 is an L^∞function.
(**A1**) $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2}|\mu_2(p)|dp\le\mu_1 \tag{12}$$

In order to deal with the disributed delay feedback term, motivated by [17], let use introduce a new dependent variable

$$y(x,\tau,p,t)=\varphi_t(x,t-p\tau), \tag{13}$$

Using (13), we have

$$\begin{cases} py_t(x,\tau,p,t)=-y_\tau(x,\tau,p,t) \\ y(x,0,p,t)=\varphi_t(x,t). \end{cases} \tag{14}$$

Thus, the problem is equivalent to

$$\begin{cases} \rho_1\varphi_{tt}-\beta(\varphi_x+\psi)_x+\mu_1\varphi_t+\displaystyle\int_{\tau_1}^{\tau_2}|\mu_2(p)|y(x,1,p,t)\,dp=0 \\ -\rho_2\varphi_{ttx}-b\psi_{xx}+\beta(\varphi_x+\psi)=0 \\ py_t(x,\tau,p,t)+y_\tau(x,\tau,p,t)=0 \end{cases} \tag{15}$$

where

$$(x,\tau,p,t)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Additionally, we consider initial conditions given by

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \varphi_{tt}(x,0)=\varphi_2(x) \\ \varphi_{ttt}(x,0)=\varphi_3(x),\ \psi(x,0)=\psi_0(x), x\in(0,1) \\ y(x,\tau,p,0)=f_0(x,-p\tau), y_t(x,\tau,p,0)=f_1(x,-p\tau),\ in\ \ (0,1)\times(0,1)\times(0,\tau_2), \\ y_{tt}(x,\tau,p,0)=f_2(x,-p\tau),\ in\ \ (0,1)\times(0,1)\times(0,\tau_2) \end{cases} \tag{16}$$

where $\varphi_0,\varphi_1,\varphi_2,\psi_0,f_0,f_1$, are given functions, and boundary conditions of Dirichlet-Dirichlet given by

$$\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\ t>0 \tag{17}$$

Next we say that the global well-posedness of problem (15)-(17) given in the following theorem.

Theorem 2.1. *Assume the assumption (12) holds. If the initial data* $(\varphi_0,\varphi_1,\varphi_2,\varphi_3,\psi_0)$ *is in* $(H_0^1(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1)\times H_0^1(0,1))$, $f_0,f_1,f_2\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))$, *then problem (15)-17) has a weak solution such that*

$$\varphi\in C(\mathbb{R}_+,H_0^1(0,1))\cap C^1(\mathbb{R}_+,L^2(0,1)),\ \psi\in C(\mathbb{R}_+,H_0^1(0,1))$$

$$\varphi_t, \varphi_{tt} \in C(\mathbb{R}_+, L^2(0,1)).$$

In addition, we have that the solution $(\varphi, \varphi_t, \varphi_{tt}, \psi)$ depends continuously on the initial data in $H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times H_0^1(0,1)$. In particular, problem (15)-17) has a unique weak solution.

2.1. **The Global Well-Posedness.** In this subsection, we will prove the global existence and the uniqueness of the solution of problem (9)-(17) by using the classical Faedo-Galerkin approximations along with some priori estimates. We only prove the existence of solution in (i). For the existence of stronger solution in (ii), we can use the same method as in (i) and one can refer to Andrade e al. [6] and Jorge Silva and Ma [13] and Feng [12].

2.1.1. *Approximate Problem.* which satisfy the following approximate problem:

$$\begin{aligned}
&\rho_1(\varphi_{mtt}, u_j) + \beta((\varphi_{mx} + \psi_j), u_{mx}) + \mu_1(\varphi_{mt}, u_j) \\
&\qquad\qquad + (\int_{\tau_1}^{\tau_2} |\mu_2(p) y_m(x,1,p,t)\, dp, u_j) = 0, \\
&b(\psi_{mx}, \theta_{jx}) + \rho_2(\varphi_{mtt}, \theta_{jx}) + \beta((\varphi_{mx} + \psi_j), \theta_{mj}) = 0 \\
&(p y_{mt}(x,\tau,p,t), \phi_j) + (y_{m\tau}(x,\tau,p,t), \phi_j) = 0 \\
&(p y_{mtt}(x,\tau,p,t), \phi_j) + (y_{m\tau t}(x,\tau,p,t), \phi_j) = 0 \qquad (18)
\end{aligned}$$

with initial conditions

$$\begin{aligned}
&\varphi_m(0) = \varphi_0^m, \varphi_{mt}(0) = \varphi_1^m, \varphi_{mtt}(0) = \varphi_2^m \\
&\varphi_{mttt}(0) = \varphi_3^m, \psi_m(0) = \psi_0^m, \psi_{mt}(0) = \psi_1^m, \\
&y_m(0) = y_0^m, y_{mt}(0) = y_1^m, y_{mtt}(0) = y_2^m \qquad (19)
\end{aligned}$$

which satisfies

$$\begin{aligned}
&\varphi_0^m \to \varphi_0, \ \ \textit{strongly in} \ \ H_0^1(0,1) \\
&\varphi_1^m \to \varphi_1, \ \ \textit{strongly in} \ \ L^2(0,1) \\
&\varphi_2^m \to \varphi_2, \ \ \textit{strongly in} \ \ L^2(0,1) \\
&\varphi_3^m \to \varphi_3, \ \ \textit{strongly in} \ \ L^2(0,1) \\
&\psi_0^m \to \psi_0, \ \ \textit{strongly in} \ \ H_0^1(0,1) \\
&\psi_1^m \to \psi_1, \ \ \textit{strongly in} \ \ H_0^1(0,1) \\
&y_0^m \to y_0, \ \ \textit{strongly in} \ \ L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\
&y_1^m \to y_1, \ \ \textit{strongly in} \ \ L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \qquad (20) \\
&y_2^m \to y_2, \ \ \textit{strongly in} \ \ L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \qquad (21)
\end{aligned}$$

By using standard ordinary differential equations theory, the problem (18)-(19) has a solution $(g_{jm}, h_{jm}, f_{jm})_{j=1,m}$ defined on $[0, t_m)$. The following estimate will give the local solution being extended to $[0,T]$, for any given $T > 0$.

2.1.2. *A Priori Estimate I.* It follows from (12), and (**??**) that

$$\begin{aligned}
&\int_0^1 \varphi_{mt}^2 dx + \int_0^1 \varphi_{mtt}^2 dx + \rho_2 \int_0^1 \varphi_{mtx}^2 dx + \int_0^1 (\varphi_{mx} + \psi_m)^2 dx \\
&\qquad + \int_0^1 \psi_{mx}^2 dx + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_m^2(x,\tau,p,t) dp d\tau dx
\end{aligned}$$

$$+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,\tau,p,t)dpd\tau dx \le C \tag{22}$$

Thus we can obtain $t_m = T$, for all $T > 0$.

2.1.3. *A Priori Estimate II.* where

$$\begin{aligned}\mathcal{G}_m(t) &= \frac{1}{2}\Big[\rho_1\int_0^1 \varphi_{mtt}^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \varphi_{mttt}^2 dx + \rho_2\int_0^1 \varphi_{mttx}^2 dx \\ &\quad +\beta\int_0^1(\varphi_{mxt}+\psi_{mt})^2dx + b\int_0^1 \psi_{mxt}^2 dx\Big] \\ &\quad +\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,\tau,p,t)dpd\tau dx \\ &\quad +\frac{\rho_2}{2\beta}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mtt}^2(x,\tau,p,t)dpd\tau dx\end{aligned}$$

Similarly to **A Priori Estimate I**, we can get there exists a positive constant C independent on m such that

$$\mathcal{G}_m(t) \le C, \quad t \ge 0. \tag{23}$$

2.1.4. *Passage to Limit.* From (22) and (23), we conclude that for any $m \in \mathbb{N}$,

$$\begin{array}{lll} \varphi_m & \textit{weakly star in} & L^2(\mathbb{R}_+, H_0^1) \\ \varphi_{mt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2) \\ \varphi_{mtt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2) \\ \psi_m & \textit{weakly star in} & L^2(\mathbb{R}_+, H_0^1) \\ \psi_{mt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2) \\ y_m & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ y_{mt} & \textit{weakly star in} & L^2(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \end{array} \tag{24}$$

By (24), we can also deduce that φ_m, ψ_m is bounded in $L^2(\mathbb{R}_+, H_0^1)$ and $\varphi_{mt}, \varphi_{mtt}$ is bounded in $L^2(\mathbb{R}_+, L^2)$. Then from Aubin-Lions theorem [15], we infer that for and, $T > 0$,

$$\begin{array}{lll} \varphi_m & \textit{strongly in} & L^\infty(0,T,H_0^1(0,1)) \\ \psi_m & \textit{strongly in} & L^\infty(0,T,H_0^1(0,1)) \end{array} \tag{25}$$

We also obtain by Lemma 1.4 in Kim [14] that

$$\begin{array}{lll} \varphi_m & \textit{strongly in} & C(0,T,H_0^1(0,1)) \\ \psi_m & \textit{strongly in} & C(0,T,H_0^1(0,1)) \end{array} \tag{26}$$

Then we can pass to limit the approximate problem (18)-(19) in order to get a weak solution of problem (15)-(17).

2.1.5. *Continuous Dependence and Uniqueness.* Firstly we prove the continuous dependence and uniqueness for stronger solutions of problem (15)-(17).
Let $(\varphi, \varphi_t, \varphi_{tt}, \varphi, \Upsilon, \Upsilon_t)$, and $(\Gamma, \Gamma_t, \Gamma_{tt}, \Xi, \Pi, \Pi_t)$ be two global solutions of problem (15)-(17) with respect to initial data $(\varphi_0, \varphi_1, \varphi_2, \varphi_0, \Theta_0, \Theta_1)$, and $(\Gamma_0, \Gamma_1, \Gamma_0, \Xi_0, \Phi_0, \Phi_1)$ respectively. Let

$$\Lambda(t) = \varphi - \Gamma$$

$$\begin{aligned}\Sigma(t) &= \varphi - \Xi \\ \chi(t) &= \Pi - \Phi \end{aligned} \tag{27}$$

Then (Λ, Σ, χ) verifies (15)-(17), and we have

$$\begin{aligned}&\rho_1\Lambda_{tt} - \beta(\Lambda_x + \Sigma)_x + \mu_1\Lambda_t + \int_{\tau_1}^{\tau_2} |\mu_2(p)|\Lambda_t\,(x, t-p)\,dp \\ &-\rho_2\Lambda_{ttx} - b\Sigma_{xx} + \beta(\Lambda_x + \Sigma) = 0 \\ &p\chi_t(x,\tau,p,t) + \chi_\tau(x,\tau,p,t) = 0\end{aligned} \tag{28}$$

where

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\Big[\rho_1\int_0^1 \Lambda_t^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \Lambda_{tt}^2 dx + \rho_2\int_0^1 \Lambda_{tx}^2 dx + \beta\int_0^1 (\Lambda_x+\Sigma)^2 dx \\ &+ b\int_0^1 \Sigma_x^2 dx\Big] + \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|\chi^2\,(x,1,p,t)\,dpd\tau dx\end{aligned} \tag{29}$$

Applying Gronwall's inequality to (**??**), we get

$$\begin{aligned}&(\|\Lambda_t\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{tx}\|^2 + \|\Sigma_x\|^2 + \|(\Lambda_x+\Sigma)\|^2 \\ &+ \int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|\|\chi\,(x,1,p,t)\,\|^2 dpd\tau) \le e^{C_2 t}\mathcal{E}(0)\end{aligned} \tag{30}$$

This shows that solution of problem (15)-(17) depends continuously on the initial data.

2.2. **Exponential stability.** In this subsection, we will prove an exponential stability estimate for problem (15) − (17), under the assumption (12), and by using a multiplier technique.

We define the energy of solution

$$\begin{aligned}\mathcal{E}\,(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + b\psi_x^2 + \beta(\varphi_x+\psi)^2 + \frac{\rho_1\rho_2}{\beta}\varphi_{tt}^2 + \rho_2\varphi_{tx}^2\right] dx \\ &+ \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y^2\,(x,\tau,p,t)\,dpd\tau dx \\ &+ \frac{1}{2}\frac{\rho_2}{\beta}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_t^2\,(x,\tau,p,t)\,dpd\tau dx\end{aligned} \tag{31}$$

Then we have the following lemma.

Lemma 2.2. *The energy $\mathcal{E}(t)$ satisfies*

$$\begin{aligned}\mathcal{E}'\,(t) &\le -\left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(p)|dp\right)\int_0^1 \varphi_t^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(p)|dp\right)\int_0^1 \varphi_{tt}^2 dx \\ &\le -\eta_0\int_0^1 \varphi_t^2 dx - \eta_0\frac{\rho_2}{\beta}\int_0^1 \varphi_{tt}^2 dx \le 0\end{aligned} \tag{32}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(p)|dp \ge 0.$

Lemma 2.3. *The functional*

$$F_1\,(t) := -\frac{\mu_1}{2}\int_0^1 \varphi_t^2 dx - \beta\int_0^1 \varphi_{tx}\varphi_x dx \tag{33}$$

satisfies

$$\begin{aligned} F_1'(t) &\leq -\beta\int_0^1 \varphi_{tx}^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \varphi_{tt}^2 dx \\ &\quad + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t) dp dx \end{aligned} \tag{34}$$

Lemma 2.4. *The functional*

$$F_2(t) := \rho_1 \int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx + \frac{\mu_1\rho_2}{2\beta}\int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \varphi_{tx}\varphi_x dx$$

satisfies,

$$\begin{aligned} F_2(t) &\leq -\frac{\rho_1\rho_2}{2\beta}\int_0^1 \varphi_{tt}^2 dx - \frac{\beta}{2}\int_0^1 (\varphi_x+\psi)^2 dx - b\int_0^1 \psi_x^2 dx \\ &\quad + \rho_2 \int_0^1 \varphi_{tx}^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ &\quad + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t) dp dx. \end{aligned} \tag{35}$$

Lemma 2.5. *The functional*

$$\begin{aligned} F_3(t) &:= \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau dx \\ &\quad + \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau dx. \end{aligned}$$

satisfies,

$$\begin{aligned} F_3'(t) &\leq -\eta_1 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau dx + \mu_1 \int_0^1 \varphi_t^2 dx \\ &\quad -\eta_1 \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t)\, dp dx \\ &\quad -\eta_1 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau dx + \mu_1 \int_0^1 \varphi_{tt}^2 dx \\ &\quad -\eta_1 \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| y_t^2(x,1,p,t)\, dp dx \end{aligned} \tag{36}$$

where $\eta_1 > 0$.

Theorem 2.6. *Assume (A1), there exist positive constants* λ_1 *and* λ_2 *such that the energy functional (31) satisfies*

$$\mathcal{E}(t) \leq \lambda_2 e^{-\lambda_1 t}, \forall t \geq 0 \tag{37}$$

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The Study of Stability of Positive Solutions for a New Class of Hyperbolic Differential System

Abstract: This presentation deals with the existence and stability of positive solution for hyperbolic system of ($p(x)$,$q(x)$)-Laplacian system of partial differential equations using a sub and super solution according to some given boundary conditions, Our result is an extension solutions to other previous studies, which treated the stationary case, this study is new for evolutionary case of this kind of the problem.

The purpose of our presentation provides a framework for image restoration.
Keywords: Hyperbolic system, Sub-super solutions method, Positive solution, Stability.

1 Introduction

We consider the following hyperbolic system of partial differential equation: find $u(x,t)$ such that $u \in L^2(0,T,H_0^1(\Omega))$, $u_t \in L^2(0,T,L^2(\Omega))$, $u_{tt} \in L^2(0,T,L^2(\Omega))$ solution of problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta_{p(x)} u$$
$$= \lambda^{p(x)} [\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)] \text{ in } \Omega_T = \Omega \times (0,T),$$
$$\frac{\partial^2 v}{\partial t^2} - \Delta_{q(x)} v$$
$$= \lambda^{q(x)} [\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)] \text{ in } \Omega_T = \Omega \times (0,T),$$

$$\begin{cases} u = v = 0 \text{ on } \partial\Omega \times (0,T), \\ (u(.,0), v(.;0)) = (\varphi_1, \varphi_2) \text{ on } \Omega \times (t=0), \\ (u_t(x,0), v_t(x;0)) = (\varphi_3, \varphi_4) \text{ on } \Omega \times (t=0) \end{cases} \quad (1)$$

Where Ω part of $\mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $1 < p(x), q(x) \in C^1(\overline{\Omega})$ are functions with

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{\Omega} p(x) < \infty,$$

$$1 < q^- := \inf_{x\in\Omega} q(x) \le q^+ := \sup_{x\in\Omega} q(x) < \infty$$

and $\Delta_{p(x)}$ is a $p(x)-$Laplacien defined as:

$$\Delta_{p(x)} u = div(|\nabla u|^{p(x)-2} \nabla u)$$

with positive parameters $\lambda, \lambda_1, \lambda_2, \mu_1, \mu_2$, and functions f, g, h, τ are monotone in in $[0, +\infty[$ such that :

$$\lim_{u\to+\infty} f(u) = +\infty \lim_{u\to+\infty} g(u) = +\infty,$$

$$\lim_{u\to+\infty} h(u) = +\infty, \lim_{u\to+\infty} \tau(u) = +\infty,$$

satisfying some natural growth condition at $u = \infty$.

In addition, we haven't considered any sign condition on $f(0), g(0), h(0), \tau(0)$.

Where the existence of positive solution of the hyperbolic partial differential equation will be proved according to the conditions of symmetry, using super and sub solutions. The linear and nonlinear stationary equations with operators of quasilinear homogeneous type as p-Laplace operator, which can be carried out according to the standard Sobolev spaces theory of $W^{m,p}$, and thus we can find the weak solutions. The last spaces consist of functions having weak derivatives which verify some conditions of integrability. Thus, we can have the non-homogeneous case of $p(.)$-Laplace operators in this last condition. We use Sobolev spaces of the exponential variable in our standard framework, so that $L^{p(.)}(\Omega)$ is used instead of Lebesgue spaces $L^p(\Omega)$.

We denote new Sobolev space by$W^{m,p}(\Omega)$. If we replace $L^{p(.)}(\Omega)$ by $L^p(\Omega)$, the Sobolev spaces becomes $W^{m,p(.)}(\Omega)$. Several Sobolev spaces properties have been extended to spaces of Orlicz-Sobolev, particularly by O'Neill in the reference [15].

Here, in our study we consider the boundedness condition in domain Ω, because many results for p-Laplacian theory are not usually verified for the $p(.)$-Laplacian theory; for that in ([28]) the quotient

$$\lambda_{p(x)} = \inf_{u\in\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx}$$

becomes 0 generally. Then $\lambda_{p(x)}$ can be positive only for some given conditions In fact, the first eigenvalue of $p(x)$-Laplacian and its associated eigenfunction cannot exist, the existence of the positive first eigenvalue λ_p and getting its eigenfunction are very important in the p-Laplacian problem study. Therefore, the study of existence of solutions of our problems has more meaning. Many studies of the experimental side have been studied on various materials that rely on this advanced theory, as they are important in electrical fluids, which state that viscosity relates to the electric field in a certain liquid.

2 Technical assumptions and main results

We give the following notation:

Let $P(\Omega)$ be the set of the all: $p:\Omega\to[1,+\infty[$ functions measurable

$$L^{p(x)}(\Omega)=\left\{u\in P(\Omega):\int_{\Omega}|u(x)|^{p(x)}dx<\infty\right\}$$

$L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$|u|_{p(x)}=\inf\left\{\lambda>0:\int_{\Omega}\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx<1\right\}$$

and we define

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\right\}$$

the norm on $W^{1,p(x)}(\Omega)$ is

$$\|u\|_{p(x)}=\inf\left\{\lambda>0:\int_{\Omega}\frac{|u|^{p(x)}+|\nabla u|}{\lambda^{p(x)}}^{p(x)}dx<1\right\}$$

we denote $W_0^{1.p(x)}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1.p(x)}$.

To explicate our ultimate approach to the combine with the system (1,1), we consider the following hyperbolic problem

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-\Delta_{p(x)}u\\ =\lambda^{p(x)}[\lambda_1 a(x)f(u)+\mu_1 c(x)h(u)] \text{ in } \Omega_T=\Omega\times(0,T),\\ u=0, \text{ on } \partial\Omega.\times(0,t),\\ u(.,0)=\varphi_1 \text{ on } \Omega\times(t=0).\end{cases}\tag{2}$$

Theorem 1. *There exists a weak positive solution of problem 2 on the interval $[0,T]$ for the fixed time $T>0$.*

Proof. We pose for a function $u^n:[0,T]\to H_0^1(\Omega)$ of the form $u^n=\sum_{i=1}^{n}d_n^i w_i$ Here we hope to select the coefficients $d_n^i(t)$ such that

$$\begin{cases}\frac{d^2}{dt^2}(u^n,w_i)+B[u^n,w_i,t]\\ =(F(u^n),w_i)\\ d_i'(0)=\varphi_1\end{cases}$$

taking

$$v=\sum_{i=1}^{n}v_i w_i(x)$$

thus

$$(u^n,v)=\sum_{i=1}^{n}d_n^i(t)v_i$$

then, we have

$$\begin{cases}\frac{d^2}{dt^2}(u^n,v)+B[u^n,v,t]\\ =(F(u^n),v)\\ (u^n,v)_{t=o}=(\varphi_1,v),\end{cases}$$

then for $i=1,2,..n$, hence,

$$\begin{cases}\left(\sum_{i=1}^{n}\frac{d^2}{dt^2}d_n^i(t)+\sum_{i,j=1}^{n}k_{i,j}d_n^i(t)\right)v_i\\ =\left(F\left(\sum_{i=1}^{n}d_n^i w_i\right),w_j\right)v_i \text{ for all } v_i\\ (u^n,v)_{t=o}=\sum_{i=1}^{n}\varphi_1 v_i\end{cases}$$

then the system 2 becomes

$$\begin{cases}\left(\sum_{i=1}^{n}\frac{d^2}{dt^2}d_n^i(t)+\sum_{i,j=1}^{n}k_{i,j}d_n^i(t)\right)\\ =\left(F\left(\sum_{i=1}^{n}d_n^i w_i\right),w_j\right) \text{ for all } v_i,\\ \sum_{i=1}^{n}d_n^{i'}(0)=\sum_{i=1}^{n}\varphi_1 w_i\end{cases}$$

We choose $n\geq N$, multiply 2 by $d_n^{i'}(t)$ sum $i=1,2,..N$ and than integrate with respect to t, we find

for all function $v\in L^{p(x)}(0,T,H_0^1(\Omega))$

and

$$\int_0^T-\langle v,u\rangle+B[u,v,t]dt$$

$$=\int_0^T(F(u),v)dt+(u(0),v(0))$$

$$=\int_0^T(F(u),v)dt+(\varphi_1,v(0))$$

since $u^n(0)\to\varphi_1$ in $L^2(\Omega)$, as $v(0)$ is arbitrary, we deduce $u(0)=\varphi_1$

This approach is based on the method of sub-super solutions which are defined as follows:

Definition 1. *A pair of nonnegative functions $\underline{u}$, $\overline{u}$ $\in L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right)\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)$ are called order sub-super solutions of (2) if $\overline{u}\geq\underline{u}$ and if $\overline{u}$ satisfies:*

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-\Delta_{p(x)}u \\ \geq\lambda^{p(x)}\left[\lambda_1 a(x)f(u)+\mu_1 c(x)h(u)\right] \text{ in } \Omega_T=\Omega\times(0,T), \\ u\geq 0, \text{ on } \partial\Omega.\times(0,t), \\ u(.,0)\geq\varphi_1 \text{ on } \Omega\times(t=0)\end{cases}$$

and $\underline{u}$ satisfies the above inequalities in reverse order .
Define $A: W^{1;p(x)}\cap C_0^+\to\left(W_0^{1;p(x)}(\Omega)\right)^*$ where

$$\langle Au,\varphi\rangle=\int_{\Omega}\left(u_{tt}-|\nabla u|^{p(x)-2}\nabla u\nabla\varphi-G(x,t,u)\nabla\varphi\right)dx,$$

where $u\in W^{1,p(x)}\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)$ and $\varphi\in W_0^{1,p(x)}(\Omega)$

Lemma 1.[Comparison principal] *Let $u,v\in W^{1,p(x)}\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)$ be positive and satisfy $Au-Av\geq 0$ in $\left(W_0^{1;p(x)}(\Omega)\right)^*$, let*

$$\varphi(x)=\min\{u(x),v(x),0\}$$

if $\varphi(x)\in W_0^{1,p(x)}(\Omega)$ (i.e. $u\geq v$ on $\partial\Omega$) than $u\geq v$ a.e in Ω.

Theorem 2. *Suppose that $(H1)-(H3)$ hold, then for every $\lambda\in[A,B]$, system (1) has at least one positive weak solution.*

Proof. For any $\upsilon=(u,v)$. Set

$$S=\begin{cases}\upsilon\in L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right)\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right): \\ \underline{\upsilon}\leq\upsilon\leq\overline{\upsilon}\end{cases}$$

and

$$S\times\overline{S}=\begin{cases}(\upsilon,\omega)\in\left(L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right)\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)\right)^2: \\ (\underline{\upsilon},\underline{\omega})\leq(\upsilon,\omega)\leq(\overline{\upsilon},\overline{\omega})\end{cases}$$

Let

$$\underline{z}^{(1)}=\underline{\omega}^{(1)}-\underline{\omega}^{(0)}$$

and

$$\underline{z}_t^{(1)}-B\underline{z}^{(1)}=F\left(.,\underline{u}^{(0)}\right)-\left[\underline{\omega}_t^{(0)}-B\underline{\omega}^{(0)}\right]$$

since

$$B\underline{z}^{(1)}=G\left(.,\overline{\upsilon}^{(0)}\right)-G\left(.,\underline{\upsilon}^{(0)}\right)\geq 0$$

that

$$\overline{\upsilon}^{(1)}\geq\underline{\upsilon}^{(1)}$$

Now, let (υ^n,ω^n) denote either $(\overline{\upsilon}^n,\overline{\omega}^n)$ or $(\underline{\upsilon}^n,\underline{\omega}^n)$.

For every n, each υ^n is a solution of the system

$$\frac{\partial^2\upsilon^n}{\partial t^2}+B(\upsilon^n)=F(x,t,\upsilon^n) \text{ in } \Omega_T$$

$$\upsilon(.,0)=G_i\left(x,t,\upsilon^{n-1}\right)$$

$$\upsilon(x,0)=\varphi_j, j=3,4 \text{ in } \Omega.$$

Furthermore, since the sequence $\{u^n\}$ is uniformly bounded in $L^{p(x)}\left(0,T,W_0^{\alpha,p(x)}\left(\overline{\Omega}_T\right)\right)$, there exist positive constants M and σ independent of n such that

$$|u^n|_{L^{p(x)}\left(0,T,W_0^{\sigma,p(x)}\left(\overline{\Omega}_T\right)\right)}\leq M,$$

we first show that the limit u^* of u^n satisfies the equation of (2) in Ω_T,

we define the operator

$$L^n\upsilon=\frac{\partial^2\upsilon^n}{\partial t^2}+B(\upsilon^n)\equiv\frac{\partial^2\upsilon^n}{\partial t^2}-|\nabla\upsilon|^{p-2}\nabla\upsilon,$$

$$F(x,t)=F\left(x,t,\upsilon^{n-1}\right)-\varphi_i\left(x,0,\upsilon^{n-1}\right)$$

this proves that $\overline{u}$ and $\underline{u}$ are both solutions of (1).

4 Some stability results of hyperbolic systems (1)

In this section, we give some a priori estimates for discrete weak solution $(u^{n,m},v^{n,m})_{0\leq n\leq N}$ which we use later to derive convergence results for the Euler scheme.

Theorem 4. *Let $(H_1),(H_2)$, and (H_3) be satisfied. Then, there exists a positive constant $C(u_0,a,b,c,d)$ depending on the data but not on N such that for all $n=1,2,3,...,N$, we have*

$$\|u^{n,m}\|_2^2 \leq C(u_0,a,b,c,d),$$

Proof. Let $k>0$ and $1\leq n\leq N$, we take $\varphi=\psi=|U_n|^k U_n$ as test function in equality (4) and (5) and we obtain

$$\int_\Omega |u^{n,m}|^{k+2}dx$$

$$+\tau^2\int_\Omega |\nabla u^{n,m}|^{p(x)-2}\nabla u^{n,m}.\nabla(|u^{n,m}|^k u^{n,m})dx =$$

$$\tau^2\left[\begin{array}{l}\frac{\int_\Omega \lambda^{p(x)}[\lambda_1 a(x)f(v^{n,m})+\mu_1 c(x)h(U_n)](|u^{n,m}|^k u^{n,m})dx}{2}\\ +\frac{\int_\Omega u^{n+1,m}+u^{n-1,m}(|U_n|^k U_n)}{2}dx\end{array}\right] \quad (9)$$

This implies that

$$\begin{aligned}\|u^{n,m}\|_{k+2}^{k+2} &\leq \tau^2 c_1\|u^{n,m}\|_{k+1}^{k+1}\\ &+\|u^{n,m}\|_{k+2}\|u^{n,m}\|_{k+2}^{k+1}\end{aligned} \quad (10)$$

If $\|u^{n,m}\|_{k+2}=0$, we get immediately the result

If $\|u^{n,m}\|_{k+2}\neq 0$, inequality (10) becomes $\|u^{n,m}\|_{k+2}\leq \tau^2 c_1+\|u^{n,m}\|_{k+2}$

$\leq Tc_2+\|u^{0,0}\|_{\infty.}$

Taking the limit as $k\to\infty$, we deduce the result .

Theorem 5. *Let hypotheses $(H_1),(H_2)$ and (H_3) be satisfied. Then, there exists a positive constant $C(u_0,a,b,c,d)$ depending on the data but not on N such that for all $n=1,2,3,...N$, we have*

$$\tau^2\sum_{j=1}^n\int_\Omega |\nabla v|^{q(x)-2}\nabla v.\nabla u_j dx$$

$$+\tau\sum_{j=1}^n\left\|\lambda^{q(x)}\lambda_2 b(x)g(u_j)\right\|_1$$

$$\leq C(u_0,a,b,c,d),$$

Proof. Let $k>0$, we define the following function

$$T_\tau(s)=\begin{cases} s \text{ if } |s|\leq k,\\ k\, sign(s) \text{ if } |s|>k\end{cases}$$

Where

$$sign(s)=\begin{cases}1 \text{ if } s>0,\\ 0 \text{ if } s=0,\\ -1 \text{ if } s<0.\end{cases}$$

In equality (4) and (5), we take $\varphi=\psi=T_k(U_i)$ as a test function and dividing this equality by k, taking limits when k goes to 0, we get

$$\begin{aligned}&\|u_j\|_1+\left\|\lambda^{p(x)}\mu_1 c(x)h(u_j)\right\|_1\\ &+\lim_{k\to\infty}\frac{\tau^2}{k}\int_{\{U_j\leq k\}}|\nabla u_j|^{p(x)}dx\\ &\leq \tau\left\|\lambda^{p(x)}\lambda_1 a(x)f(v)\right\|_1+\|u_{j-1}\|_1\end{aligned} \quad (11)$$

And respectively,

$$\|u_j\|_1+\left\|\lambda^{q(x)}\lambda_2 b(x)g(u_j)\right\|_1$$

$$+\lim_{k\to\infty}\frac{\tau^2}{k}\sum_{j=1}^n\int_{\{U_j\leq k\}}|\nabla v|^{q(x)-2}\nabla v.\nabla u_j dx$$

$$\leq \|u_j\|_1+\left\|\lambda^{q(x)}\lambda_2 b(x)g(u_j)\right\|_1$$

$$+\left\||\nabla v|^{q(x)-2}\nabla v\right\|^q.\lim_{k\to\infty}\frac{\tau^2}{k}\sum_{j=1}^n\int_{\{U_j\leq k\}}|\nabla u|^{q(x)}dx,$$

Which implies

$$\|u_j\|_1+\left\|\lambda^{q(x)}\lambda_2 b(x)g(u_j)\right\|_1$$

$$\begin{aligned}&+\left\||\nabla v|^{q(x)-2}\nabla v\right\|^q.\lim_{k\to\infty}\frac{\tau^2}{k}\sum_{j=1}^n\int_{\{U_j\leq k\}}|\nabla u_j|^{q(x)}dx\\ &\leq \tau^2\left\|\lambda^{q(x)}\mu_2 d(x)\tau(v)\right\|_1+\|u_{j-1}\|_1\end{aligned} \quad (12)$$

$$\leq \tau^2\left[\left\|\lambda^{q(x)}\lambda_2 b(x)g(u_j)\right\|_1+\left\|\lambda^{q(x)}\mu_2 d(x)\tau(v)\right\|_1\right]$$

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L'EXISTENCE ET L'UNICITÉ DE PROBLÈME D'ÉVOLUTION NON CLASSIQUE

ATMANIA ISLAH[1] AND ZITOUNI SALAH[2]

Abstract. Le but de ce travail est d'étudiée quelques problème mixtes non locaux par la méthode des inégalités énergétiques(estimation à priori). ce problème est l'équation hyperbolique de deuxième degré en combinant une condition classique et une autre intégrale définies comme suivant

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x}) = f(x,t) \text{ dans } Q = \Omega \times [0,T],$$

avec le cindition de Newmann

$$u_x(0,t) = 0,$$

La condition intégrale

$$\int_0^\ell u dx = 0,$$

les conditions initials

$$\ell_1 u = u(x,0) = \varphi(x), \quad \ell_2 u = u_t(x,0) = \psi(x),$$

Où $\Omega = [0,\ell]$, et $a(x,t)$ est une fonction vérifiée les conditions suivants

$$c_0 \leq a \leq c_1,\ \frac{\partial a}{\partial t} \leq c_2,\ \frac{\partial a}{\partial x} \leq c_3, \frac{\partial^2 a}{\partial t^2} \leq c_4,\ \frac{\partial^2 a}{\partial x \partial t} \leq c_5,$$

et $c_i \ \ i = \overline{0-5}$, des constantes positives. on prouve l'existence et l'unucité de la solution du problème posée par la méthode énérgitique.

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INÉGALITÉS SUR LE RAYON NUMÉRIQUE D'UN OPÉRATEUR BORNÉ

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Abstract. Soit H un espace de Hilbert complexe et A un opérateur linéaire borné sur H. Le rayon numérique de A est le réel positif défini par

$$w(A) = \sup \{ |\langle Ax, x \rangle| \, ; x \in H, \|x\| = 1 \} .$$

Dans ce travail on a étudié les inégalités du rayon numérique d'un opérateur borné sur un espace de Hilbert.

On a donné au début une étude initiative sur l'image et le rayon numériques. Cette étude concerne les propriétés de base de ces concepts dont la plus importante pour ce travail est que le rayon numérique définisse une norme équivalente à la norme usuelle.

Dans une partie de ce travail, on a présenté des différents types d'inégalités pour un opérateur: des inégalités générales, des inégalités de puissance et des inégalités en sens inverse.

Au dernière partie, on a étudié des inégalités pour deux opérateurs, parmi ces inégalités, des inégalités de base pour le produit et des inégalités pour un opérateur inversible.

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Polynômes polaires associés aux polynômes orthogonaux sur le cercle unité

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RESUME

Soit μ une mesure positive finie définie sur la tribu borélienne de $\mathbb{C}$ et concentré sur le cercle unité $T = \{z, \ |z| = 1\}$. μ est absolument continue par rapport à la mesure de Lebesgue $d\theta$ sur $[-\pi, +\pi]$, i.e.

$$d\mu(\theta) = \rho(\theta)d\theta, \ \rho \geq 0, \ \rho \in L^1([-\pi, +\pi], d\theta). \tag{1.1}$$

Soit $\{L_n\}$ le système de polynômes orthogonaux sur le cercle unité in brief **POCU** i.e. $L_n(z)$ vérifie les relations d'orthogonalité suivantes:

$$\int_0^{2\pi} L_n(z)(\overline{z})^k \rho(\theta)d\theta = 0, \ k = 0, 1, ..., n-1, \quad z = e^{i\theta}. \tag{1.2}$$

Soit $\alpha \in \mathbb{C}$ fixé, que nous appelerons pôle, définissons les deux suites des polynômes polaires respectivement de premier ordre et de seconde ordre notés $P_n(z)$, $Q_n(z)$ associés aux polynômes orthogonaux sur le cercle unité (POCU) $L_n(z)$, comme suit. $P_n(z) = z^n + ..., Q_n(z) = z^n + ...,$

$P_n(z)$ est un polynôme unitaire (monique)) est solution de l'équation différentielle suivante:

$$(n+1) L_n(z) = ((z-\alpha) P_n(z))' \tag{1.3}$$

$Q_n(z)$ est un polynôme unitaire (monique)) est solution de l'équation différentielle suivante:

$$(n+2)(n+1) L_n(z) = \left((z-\alpha)^2 Q_n(z)\right)'' \tag{1.4}$$

On étudie dans cet manuscript les propriétés structurelles des polynômes polaires P_n,et Q_n de premier et seconde ordre respectiveent.

Pour assurer l'unicité des polynômes polaires P_n,et Q_n de premier et seconde ordre évidemment on doit supposer les deux conditions :

$$[((z-\alpha)\,P_n\,(z))]_{z=\alpha} = 0$$

et

$$\left[\left((z-\alpha)^2\,Q_n\,(z)\right)'\right]_{z=\alpha} = 0$$

1 Définition et propriétes générales des polynômes orthogonaux et polynômes associés sur le cercle unité

Notons que $\Phi_n(z) = (z-\alpha)\,P_n\,(z)$ est un polynôme monique, primitive de $(n+1)\,L_n\,(z)$, normalisé par la relation $\Phi_n(\alpha) = 0$. Une conséquence directe des relations précédentes est que $P_n\,(z)$ et $Q_n\,(z)$ satisfaitent les relations d'orthogonalité suivantes:

$$\int_0^{2\pi} ((z-\alpha)\,P_n\,(z))'\,(\overline{z})^k\,\rho(\theta)d\theta = 0,\;\; k = 0,1,...,n-1, \quad \left(z = e^{i\theta}\right). \tag{1.5}$$

et

$$\int_0^{2\pi} \left((z-\alpha)^2\,Q_n\,(z)\right)''\,(\overline{z})^k\,\rho(\theta)d\theta = 0,\;\; k = 0,1,...,n-1, \quad \left(z = e^{i\theta}\right) \tag{1.6}$$

Ce type d'orthogonalité généré par des opérateurs differentiels a été introduit initialement par Aptekarev,Cachafeiro,Marcellán,(voir: [2]) et Alfaro et Marcellán,(voir:14).Une étude similaire a été effectuée par A.Fundora, H.Pijeira et W.Urbina , (voir:[22,49]),dans le cas où la mesure est concentrée sur le segment $[-1,+1]$, au lieu du cercle $T = \{z,\; |z| = 1\}$.Une étude autre similaire a été effectuée par Ya.Laskri ,A.Rehouma (voir:[26,27,28]) ,dans le cas où la mesure de surface (planar measure) est concentrée sur le disque unité (polynômes polaires des polynômes orthogonaux de type de **Bergman**).En englais (Polar Bergman polynomials on domains with corners) (voir:[26,27,28]).

Le chapitre I de ce manuscrit est consacré à l'étude des propriétés fondamentales des polynômes orthogonaux associés à une mesure concentrée sur le cercle unité,**POCU**. On définit dans ce chapitre aussi les polynômes polaires associés aux polynômes orthogonaux sur le cercle unité. On donne quelques propriétés générales de ces polynômes.

Le chapitre II est consacré à l'étude du comportement asymptotique des polynômes orthogonaux associés à une mesure concentrée sur le cercle unité,(**POCU**).

le chapitre III de ce manuscrit contient une partie importante des résultats originaux de cet exposé. Il est consacré à l'étude du comportement asymptotique des polynômes polaires associés aux polynômes orthogonaux sur le cercle unité,(**POCU)**. On y trouve aussi quelques notions et résultats concernant la localisation des zéros des polynômes polaires.

1.1 Généralités sur les polynômes orthogonaux sur le cercle unité

Soit μ une mesure positive, finie, non discrète et définie sur la tribu Borélienne $B(\mathbb{C})$ de $\mathbb{C}$, $\mathbb{C}$ étant muni de sa topologie usuelle. μ est supportée sur le cercle unité $\partial D = T = \{z, |z| = 1\}$. On suppose dans tout ce qui suit que, (voir:[47,21,40,55,57])

$$z^n \in L^2(\mu, \mathbb{C})\,; \quad n = 0, 1, 2, ...$$

Notons par $\{L_n(z)\}_{n=0}^{\infty}$ la suite des polynômes orthogonaux sur le cercle unité T par rapport à la mesure μ, $\{L_n(z)\}_{n=0}^{\infty}$ vérifient donc les relations d'orthogonalité suivantes

$$\frac{1}{2\pi}\int_0^{2\pi} L_n(z)\overline{L_m(z)}d\mu(\theta) = \delta_{n,m}\|L_n\|^2_{L^2(\partial D, d\mu)} \quad , \ n, m \in \mathbb{N}. \qquad \left(z = e^{i\theta}\right) \tag{1.7}$$

et

$$\|L_n\| = \|L_n\|_{L^2(\partial D, d\mu)} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} |L_n(z)|^2 \, d\mu(\theta)} \quad , n = 0, 1, 2..... \qquad \left(z = e^{i\theta}\right)$$

Pour l'unicité, posons

$$L_n(z) = z^n + \xi_{n,1}z^{n-1}.... + \xi_{n,n-1}z + \xi_{n,n}$$

Remarquons que

$$\xi_{n,n-1} = L_n'(0) \qquad \text{et} \qquad \xi_{n,n} = L_n(0)$$

Notons par **OPUC** l'ensemble des polynômes orthogonaux sur le cercle unité.(in short abbreviation **O**rthogonal **P**olynomials on the **U**nit **C**ircle).

Définition 1.1.*Soit $\{L_n(z)\}_{n=0}^{\infty}$ le système de polynômes orthogonaux associés à la mesure μ sur le cercle unité. On appelle polynômes réciproques des polynômes les polynômes notés $\{L_n^*(z)\}$ et reliés aux polynômes orthogonaux $\{L_n(z)\}$ par la relation*

$$L_n^*(z) = z^n\overline{L_n\left(\frac{1}{\overline{z}}\right)} \tag{1.8}$$

Donnons dans le théorème suivant quelques propriétés de ces polynômes

Théorème 1.1. (voir :[47,21,40,55,57])
. *Les polynômes* $\{L_n^*(z)\}$ *et* $\{L_n(z)\}$ *vérifient les relations suivantes*

$$L_{n+1}(z) = zL_n(z) - \overline{\alpha_n} L_{n+1}(0) L_n^*(z) \tag{1.9}$$

où $\alpha_n = -\overline{L_{n+1}(0)}$,.α_n s'appellent les constantes des Verblansky, (voir:[47,21,40,55,57,59]),Et

$$L_n^*(z) = \overline{L_n(0)} z^n + \overline{L_n'(0)} z^{n-1} ......\overline{\xi_{n,n}} z + 1$$

encore

$$\|L_{n+1}\|^2_{L^2(\partial D, d\mu)} = \left(1 - |L_{n+1}(0)|^2\right) \|L_n\|^2_{L^2(\partial D, d\mu)} = \tag{1.10}$$

et,. (voir :[47,21,40,55,57])

$$\|L_{n+1}\|^2_{L^2(\partial D, d\mu)} = \prod_{k=0}^{n} \left(1 - |\alpha_k|^2\right)$$

Aussi, on a

$$\langle L_n, z^n \rangle_{L^2(\partial D, d\mu)} = \|L_n\|^2_{L^2(\partial D, d\mu)} \tag{1.11}$$

dans le même contexte

$$\left\langle L_n(z), \frac{1}{z} \right\rangle_{L^2(\partial D, d\mu)} = L_{n+1}(0) \langle L_n^*(z), 1 \rangle_{L^2(\partial D, d\mu)}$$

et

$$L_n^*(z) = L_{n-1}^*(z) + z\overline{L_{n,}(0)} L_{n-1}(z) \tag{1.12}$$

par suite

$$\left| \frac{L_{n+1}(z)}{L_n(z)} - z \right| = \left| \frac{L_{n+1}^*(z)}{L_n^*(z)} - 1 \right| = |L_{n+1}(0)| \qquad \text{, pour } |z| = 1.$$

et

$$L_n^*(z) = 1 + \sum_{k=0}^{n-1} \overline{L_{k+1}(0)} L_k(z).$$

Si, (voir:[56,59,55,57])

$$d\mu = \rho(\theta) \frac{d\theta}{2\pi} + d\mu_s$$

$\rho(\theta)$ est une fonction poids absolument continue et positive sur $[0 \quad 2\pi]$, $d\mu_s$ est une mesure singulière.La formule de **Szegö**, (voir :[55,56,55,57,59])

$$\prod_{n=0}^{\infty} \left(1 - |\alpha_n|^2\right) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log(\rho(\theta))\, d\theta \right)$$

1.2 Polynômes noyaux

Définition 1.2 *Soit* $\{L_n(z)\}_{n=0}^{\infty}$ *le système de polynômes orthogonaux associés à la mesure* μ *sur le cercle unité. On appele polynômes noyaux associés aux polynomes* $\{L_n(z)\}_{n=0}^{\infty}$ la suite des polynômes notés $\{k_n(z,w)\}_{n=0}^{\infty}$ définies par la relation suivante:

$$k_n(z,w) = \sum_{k=0}^{n} \frac{\overline{L_k(w)}L_k(z)}{\|L_k\|^2} \tag{1.13}$$

Ceci étant, on obtient la relation dite de de **Christoffel-Darboux** suivante:

Lemme 1.1 (lemme de **Christoffel-Darboux,** (voir:[14,23,43,44,57,59])

$$k_n(z,w) = \frac{L_{n+1}^*(z)\, L_{n+1}^*(w) - L_{n+1}(z)\, L_{n+1}(w)}{\|L_{n+1}\|^2 (1 - z\overline{w})} \tag{1.14}$$

Définition 1.3 Les polynômes orthogonaux noyaux à l'origine sont définis par

$$k_n(z,0) = \sum_{k=0}^{n} \frac{\overline{L_k(0)}L_k(z)}{\|L_k\|^2} \tag{1.15}$$

Ces polynômes sont orthogonaux sur le cercle unité par rapport à la même mesure,(voir:[4,14,23,43,44,57,59]).

Parmis les propriétés des polynômes orthogonaux noyaux à l'origine, nous avons le lemme suivant:

Lemme 1.3 (voir:[4,14,23,43,55])

$$k_n(z,0) = \frac{L_n^*(z)}{\|L_n\|_\mu^2}$$

si Q_n est un polynôme de degré $\leq n$ alors $k_n(.,0)$ satisfait la relation suivante:

$$\int_{|z|=1} \overline{\overline{D}^s k_n(z,0)} Q_n(z)\, d\mu(\theta) = Q_n^{(s)}(0) \tag{1.16}$$

Et

$$k_n'(z,0) = \frac{L_n^{*\prime}(z)}{\|L_n\|_\mu^2}$$

i.e

$$zk_n'(z,0) = \frac{nL_n^*(z) - (L_n'(z))^*}{\|L_n\|_\mu^2}$$

1.3 Propriétés générales des polynômes polaires primitives des POCU

L'integration de deux propriétés structurelles (1.3) ,(1.4) implique que les polynômes polaires P_n ,Q_n vérifient les relations suivantes:

$$P_n(z) = (n+1)\frac{\int\limits_{\alpha}^{z} L_n(t)dt}{z-\alpha} \qquad ,(z \neq \alpha), n = 1,2,3... \tag{1.17}$$

avec la condition

$$[(z-\alpha)P_n(z)]_{z=\alpha} = 0$$

et

$$Q_n(z) = (n+1)(n+2)\frac{\int\limits_{\xi}^{z}\int\limits_{\xi}^{t} L_n(u)\,dudt}{(z-\alpha)^2}, \qquad (z \neq \alpha),, n = 1,2,3.. \tag{1.18}$$

avec la condition

$$\left[(z-\alpha)^2 Q_n(z)\right]'_{z=\alpha} = 0$$

Le pole α est regulier pour les deux polynômes polaires P_n ,Q_n.En effet d'apres la régle de l'Hôpital , (1.17) et (1.18) impliquent

$$\lim_{z\to\alpha} P_n(z) = \lim_{z\to\xi}\frac{(n+1)\int\limits_{\xi}^{z} L_n(t)dt}{z-\xi} = (n+1)L_n(\alpha) \tag{1.19}$$

et

$$Lim_{z\longrightarrow\alpha} Q_n(z) = \frac{(n+1)^2(n+2)}{2}L_n(\alpha) \tag{1.20}$$

2 Comportement asymptotique des polynômes orthogonaux et leurs polynomes reciproques

Le problème du comportement asymptotique des polynômes orthogonaux sur le cercle unité a intéressé différents auteurs dans divers contextes. Ce problème a été étudié de façon approfondie par Szegö ((voir:[56,55]), 1921), Krein (voir:[3]), 1945), Geronimus(voir:[23,26]), 1958), Nevai, ((voir:[4]), ,1979,et ((voir:[5]) , 1997), Li-Chien Shen ((voir:[26]), 2000) et Lubinsky ((voir:[16]), 2007).

2.1 Comportement asymptotique des polynômes orthogonaux sur le cercle unité

Notons par $\{L_n(z)\}$ la suite des polynômes orthogonaux associés au cercle unité et à la mesure μ. Le comportement asymptotique des polynômes $\{L_n(z)\}$ a été étudié par les auteurs cités auparavent sous plusieurs conditions. Citons les principaux cas

a) μ est absolument continue i.e. $d\mu = \rho(\theta)d\theta$; $\rho \in L^1([0,2\pi], d\theta)$ et $\log(\rho) \in L_1([0,2\pi], d\theta)$; *$d\theta$ étant la mesure de Lebesgue sur* $[0,2\pi]$, $\rho \geq 0$

b) μ n'est pas absolument continue. La partie absolument continue de μ vérifie la condition suivante dite de Szego , (voir:[23,40,55,56,59])

$$\int_0^{2\pi} \log \mu' > -\infty \tag{2.1}$$

ou la condition dite de **Nevai** (voir:[47])

$$\mu' > 0 \ \text{ presque partout sur } \ [0,2\pi] \tag{2.2}$$

La formule asymptotique obtenue est de la forme

$$\underset{n\longrightarrow\infty}{Lim} \frac{L_n(z)}{z^n} = \frac{1}{\overline{D_\rho\left(\frac{1}{\bar{z}}\right)}} \tag{2.3}$$

uniforméments sur les compacts de $\{z, z \in \mathbb{C}, |z| > 1.\}$, $(voir: [23, 47, 55, 56, 57])$

D_ρ est dite fonction de **Szegö** et dont les propriétés se trouvent dans le théorème suivant:

Théorème 2.1 (voir:[53,18,33,55,56]).*Soit ρune fonction non négative définie sur $[-\pi,\pi]$ tel que $\rho \in L_2([-\pi,\pi], d\theta)$ et $\log(\rho) \in L_1([-\pi,\pi], d\theta)$; $d\theta$ étant la mesure de Lebesgue sur $[-\pi,\pi]$, alors la fonction suivante*

$$D_\rho(z) = \exp\left\{\frac{1}{2\pi p}\int_0^{2\pi} \log\left(\rho\left(e^{i\theta}\right)\right)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right\}, \quad (|z|<1) \tag{2.4}$$

dîte fonction de Szegö associée au domaine D et à la fonction f posséde les propriétés suivantes:

(i) $D_\rho \in H^p(D)$.

(ii) $D_\rho(z) \neq 0$; $\forall z \in D$.

(iii) $\left|D_\rho\left(e^{i\theta}\right)\right|^p = f(\xi)$; *presque partout sur Γ, où $D_f\left(e^{i\theta}\right)$ est la limite radiale de D_f.*

(vi) $D_\rho(0) > 0$.

Définition 2.1 (voir:[22,49,49,46])*Soit* Q_n *polynôme de degré* n. *Notons par* $z_{n,1}, z_{n,2}, ...z_{n,n}$ *les zéros simples de* Q_n *et par* $\nu_n(Q_n)$ la mesure suivante dite mesure de masse supportée par *les zéros de* Q_n

$$\nu_n(Q_n) = \frac{1}{n}\sum_{k=1}^{n}\delta_{z_{n,k}}$$

$\delta_{z_{n,k}}$ *est la mesure de* ***Dirac*** *concentrée sur le point* $z_{n,k}$.

Définition 2.2. Soit $\{\mu_n\}_{n=1}^{\infty}$ une suite de mesures à support compact. On dit que $\{\mu_n\}_{n=1}^{\infty}$ convege faiblement vers la mesure μ quand $n \to \infty$ si

$$\lim_{n\to\infty}\int f d\mu_n = \int f d\mu$$

pour toute fonction f continue sur $\mathbb{C}$ à support compact. Dans ce cas on écrit $\mu_n \overset{*}{\to} \mu$ ou $d\mu_n \overset{*}{\to} d\mu$ ou $d\mu_n(x) \overset{*}{\to} \mu'(x)dx$ si μ est absolument continue.

Notons par $\|.\|_{\triangle}$ la norm sup sur l'ensemble $\triangle$ et Cap($\triangle$) la capacité logarithmique de $\triangle$ ($voir$: $[22, 49, 49, 46, 37, 38, 50, 61]$),sur la notion de capacité logarithmique).Ceci étant on a le résultat suivant

Lemme 2.1 (voir:[22,49,46,37,38,59]).Soit $\Delta \subset \mathbb{C}$ un ensemble compact tel que

$$int(\triangle) = \varnothing, \ \mathbb{C} \setminus (\triangle) \text{ est connexe et } cap(\triangle) > 0$$

Soit $\{Q_n(z)\}_{n=0}^{\infty}$ une suite de polynômes moniques (i.e. $Q_n(z) = z^n + ...,$) tels que

$$\limsup_{n\to\infty}\|Q_n\|_{\Delta}^{\frac{1}{n}} \le Cap(\Delta) \ .$$

Alors

$$\nu_n(Q_n) \overset{*}{\longrightarrow} \omega_{\Delta}$$

ω_{Δ} est la mesure équilibrée de Δ (voir:[22,49,46]), pour plus de détails sur la mesure équilibrée).

Théorème 2.2 (voir:[22,49,46])

$$\lim_{n\to\infty}\frac{Q_n'(z)}{nQ_n(z)} = \lim_{n\to\infty}\int\frac{d\nu_n(x)}{z-x} = \int\frac{\lim\limits_{n\to\infty} d\nu_n(x)}{z-x} = \int\frac{d\omega_{\Delta}(x)}{z-x}. \quad (2.5)$$

où $\lim\limits_{n\to\infty}\nu_n(x) = \omega$ (au sens de la convergence faible), ν_n est la *mesure de masse supportée par* les zéros de Q_n.

Lemme 2.2(voir:[22,49,46]).Soit $\{Q_n(z)\}_{n=0}^{\infty}$ une suite de polynômes. Alors pour $j \in \mathbb{N}$ on a

$$\limsup_{n\to\infty}\left(\frac{\left\|Q_n^{(j)}\right\|_{\Delta}}{\|Q_n\|_{\Delta}}\right)^{\frac{1}{n}} \leq 1$$

Théorème 2.3 (voir:[40,47]).Soit μ une mesure de Borel finie définie sur le cercle unité (sur $[0\ \ 2\pi]$).Notons par $\{\varphi_n\}_{n\in\mathbb{N}}$ le système de plynômes orthonormés associés à la mesure μ. $\{\varphi_n\}_{n\in\mathbb{N}}$ vérifient donc les relations suivantes

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi_n(e^{i\theta})\overline{\varphi_m}(e^{i\theta})d\mu(\theta) = \delta_{mn}$$

On suppose aussi que μ vérifie la condition de **Nevai** (2.2) ou la condition de Szego (2.1). Alors on a pour tout $m \in \mathbb{N}$

$$\lim_{n\to\infty}\frac{z^m\varphi_n^{(m)}(z)}{n^m\varphi_n(z)} = 1 \quad , \qquad (z = e^{i\theta}). \tag{2.6}$$

2.2 Comportement asymptotique des polynômes dérivées

Considérons le système de polynômes orthognaux moniques $\{L_n(z)\}_{n\in\mathbb{N}}$,c-à-d tels que $(L_n(z) = z^n + ...)$ orthognaux associés à la mesure μ, μ $(d\mu(\theta) = \rho(\theta)d\theta$, voir (1.1)). On s'intéresse à la limite du rapport $\frac{L_{n+1}(z)}{L_n(z)}$. **Rahmanov** (voir:[5]) a démontré que si $\mu' > 0$ presque partout sur $[0\ \ 2\pi]$, alors on a

$$\lim_{n\to\infty}\frac{L_{n+1}(z)}{L_n(z)} = z \tag{2.7}$$

Lubinsky (voir:[40]) démontrera le résultat suivant:

Théorème 2.4 (voir:[40]) Considérons le système de polynômes orthognaux moniques $\{L_n(z)\}_{n\in\mathbb{N}}$ $(L_n(z) = z^n + ...,)$ orthognaux associés à la mesure μ. Supposons qu'on a

$$\lim_{n\to\infty}\frac{L_{n+1}(z)}{zL_n(z)} = 1 \tag{2.8}$$

pour z tel que $|z| = 1$. Alors pour $m \geq 1$ on a

$$\lim_{n\to\infty}\frac{L_{n+1}^{(m)}(z)}{zL_n^{(m)}(z)} = 1 \tag{2.9}$$

uniformément dans $\{z : |z| \geq 1\}$.

Les lemmes suivants nous seront utiles pour la suite:

Lemme 2.4 .(voir:[40]),(théorème de **Lucas,(voir:[19,28]),**(théorème de **Badkov,(voir:[12])**)

Soit P_n un polyôme de degré n. On suppose que tous les zéros de P_n sont situés dans le disque unité. Alors tous les zéros des polynômes dérivés $P_n^{(m)}$, $m = 0, 1, 2....k$, sont situés dans le disque unité. D'autre part Si $|z| = 1$,alors

$$|P_n'(z)| \geq \frac{n}{2} |P_n(z)|$$

Lemme 2.5 .(voir:[18]))(théorème de **Abdul Aziz**)

Soit $P_n = z^n + \sum_{j=0}^{n-1} a_j z^j$ un polynôme de degré n. Posons $M = \underset{0 \leq j \leq n-1}{Max} |a_j|$. Alors tous les zéros de P_n sont contenus dans le cercle $|z| \leq 1 + M$.

Lemme 2.6 .(voir:[18]))(théorème de **Abdul Aziz**)

Soit $P_n = z^n + \sum_{j=0}^{n-1} a_j z^j$ un polynôme de degré n. Posons $M = \underset{0 \leq j \leq n-1}{Max} |a_j|$, alors tous les zéros de P_n sont contenus dans la région

$$\frac{|a_0|}{2(1+M)^{n-1}(1+nM)} \leq |z| \leq 1 + \lambda_0 M$$

où λ_0 est la solution unique de l'équation:

$$x = 1 - \frac{1}{(1+Mx)^n}$$

dans l'intervalle $[0 \quad 1]$.

Lemme 2.7.(voir:[54])(théorème de **Q.I.Rahman**)

Soit $P_n(z)$ un polynôme de degré n tel que tous les zéros de $P_n(z)$ sont contenus dans le disque unité $|z| \leq 1$. Si $|a| > \frac{n+2}{n}$, alors le polynôme :

$$((z-a) P_n(z))' \tag{2.10}$$

possède une seule racine dans le disque

$$|z - a| \leq \frac{|a| + 1}{n + 1} \tag{2.11}$$

toutes les autre $n - 1$ racines sont contenues dans le disque unité $|z| \leq 1$.

Ce chapitre contient Les principaux résultats originaux de la thèse. On définira et on étudiera les propriétés algébriques et le comportement asymptotique des polynomes polaires associés aux polynômes orthogonaux sur le cercle unité.

2.3 Localisation des zeros des polynômes polaires associés aux polynômes orthogonaux sur le cercle unité

Proposition 2.1. *Les polynômes polaires P_n en les comparant avec leur polynômes orthogonaux associés $L_{n,\mu}$ vérifient*

$$P_n^{(k)}(\alpha) = \frac{n+1}{k+1} L_n^{(k)}(\alpha) \qquad , k = 0, 1, 2...n. \tag{2.12}$$

Preuve Considérons (1.3).Remarquons que

$$((z-\alpha) P_n(z))^{(k)} = (z-\alpha) P_n^{(k)}(z) + k P_n^{(k-1)}(z) = (n+1) L_n^{(k-1)}(z)$$

Posons $z = \alpha$, dans la formule précédente on obtient l'assertion (2.12).■

Le lemme suivant compare les coefficients du développement de **Taylor** de L_n et son polynôme polaire P_n.

Lemme 3.8 Si

$$L_n(z) = \sum_{k=0}^{n} A_{n,k} (z-\alpha)^k \quad \text{et} \qquad P_{n,\xi}(z) = \sum_{k=0}^{n} B_{n,k} z^k$$

alors

$$B_{n,k} = \frac{n+1}{k+1} A_{n,k}, \qquad k = 0, 1, 2...n \tag{2.13}$$

Preuve (2.13) est une conséquence de (2.12).■

Lemme 3.9 (voir:[22,49,60]) *Posons*

$$f(z) = \sum_{k=0}^{n} \alpha_{nk} C_k^n z^k, \quad \text{et} \quad g(z) = \sum_{k=0}^{n} \beta_{nk} C_k^n z^k, \quad \alpha_{nk}, \beta_{nk} \in \mathbb{C}, \tag{2.14}$$

Notons

$$h(z) = \sum_{k=0}^{n} \alpha_{nk} \beta_{nk} C_k^n z^k. \tag{2.15}$$

Supposons que tous les zéros de $f(z)$ appartiennent au disque fermé $\overline{\Delta}$. Notons par $\lambda_{n,1}, \lambda_{n,2}, ...\lambda_{n,n}$ les zéros de $g(z)$. Alors tout zéro de $h(z)$ peut s'écrire sous la forme $\lambda_{n,k}\gamma_{n,k}$,où $\gamma_{n,k} \in \overline{\Delta}$.

Dans [**?**] , **Pijeira** a montré à l'aide de ce thorème que tous les zéros des polynômes polaires dans le cas d'une mesure concentrée sur le segment (qui seront supposés moniques), appartiennent au disque fermé borné Δ=$\{z, z \in C : |z| \leq b + c|\alpha|\}$, où b, c sont deux constantes convenablement choisies.

On va maintenant énoncer un lemme dont la démonstration est un peu compliquée mais son application est la base de la formule asymptotique à l'extérieur d'un contour. Ce résultat a été obtenu par **G.Szegô** en 1921.

Lemme 3.10 (voir:[50,55,56,61]) *Soit Ω un domaine simplement connexe, $\Gamma = \partial\Omega$ est un contour de Jordan rectifiable.Soit φ la transformation conforme de Ω vers l'extérieur du cercle unité,vérifiant les conditions suivantes:*

$$\varphi(\infty) = \infty \quad et \quad \varphi'(\infty) = \tau. \tag{2.16}$$

Soient $z_k^{(n)} \in ext(\Gamma)$, $k = 1, 2...n$ et $P_n(z)$ le polynôme d'interpolation vérifiant:

$$P_n\left(z_k^{(n)}\right) = f\left(z_k^{(n)}\right), \quad k = 1, 2...n. \quad (z \in \Omega). \tag{2.17}$$

Si

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n}\left(z - z_k^{(n)}\right)}{(\tau.\varphi(z))^n} = 1. \tag{2.18}$$

Uniformément sur les compacts de Ω.

Alors

$$\lim_{n\to\infty} P_n(z) = f(z). \tag{2.19}$$

Uniformément sur les compacts de Ω.

Lemme 3.11.(voir:[59])

Soit $(L_{n,\mu})_n$ la suite des polynômes orthogonaux sur le cercle unité associés à la mesure μ. Alors les zeros de $L_{n,\mu}(z)$ sont dans le disque férmé unitaire $\overline{\Delta} = \{|z| \leq 1\}$.

Lemme 3.12.(voir:[59]). *Soit $(L_n(z,\beta))_n$ la suite des polynôme paraorthogonaux sur le cercle unité associés à la mesure μ, i-e*

$$L_{n+1}(z,\beta) = zL_n(z) + \beta L_n^*(z), \qquad , \ |\beta| = 1. \tag{2.20}$$

Alors tous les zeros de $L_n(z,\beta)$ sont sur le cercle unité: $\partial\overline{\Delta} = \{|z| = 1\}$.

Le comportement asymptotique des polynômes orthogonaux sur le cercle unité et leur polynômes polaires constitue un lieu de rencontre privilègié pour diverses disciplines des mathématiques. Ce paragraphe contient les premiers résultats originaux de cet manuscript.

3 Comportement asymptotique des polynômes polaires primitives de POCU et leurs polynômes derivés.

3.1 Comportement asymptotique des polynômes polaires primitives des POCU

Théorème.3.1 (voir:[26,27,28])*Soit* $\{L_n(z)\}_{n=0,1,2....}$ *le système de polynômes orthogonaux relativement à la mesure* μ. *On suppose que la mesure* μ *vérifie la condition de* ***Nevai*** *suivante :*

$$\mu' > 0 \text{ presque partout sur le cercle unité.}$$

Notons par $\{P_n(z)\}_{n=0,1,2....}$ *la suite des polynômes polaires unitaires de premier ordre associés à* μ. *dont le coefficient de* z^n *est égal à* $+1$. *Alors*

$$\lim_{n\to\infty} \frac{L_n(z)}{P_n(z)} = \frac{z-\alpha}{z} \tag{3.1}$$

et

$$\lim_{n\to\infty} \frac{P_{n+1}(z)}{zP_n(z)} = 1 \tag{3.2}$$

uniformément sur les compacts de $\{z, |z| \geq 1\}$.

$$Lim_{n\longrightarrow\infty} \frac{Q_n'(z)}{nQ_n(z)} = \frac{1}{z} \tag{3.3}$$

uniformément sur les compacts de $\{z, |z| \geq 1\}$.

Preuve: Rappelons que par définition les polynômes polaires $P_n(z)$ sont solutions de l'équation différentielle suivante:

$$(n+1)L_n(z) = ((z-\alpha)P_n(z))' = P_n(z) + (z-\alpha)P_n'(z).$$

Divisons les deux membres de (3.3) par $nP_n(z)$, on obtient

$$\frac{n+1}{n}\frac{L_n(z)}{P_n(z)} = \frac{1}{n} + (z-\alpha)\frac{P_n'(z)}{nP_n(z)} \tag{3.4}$$

Remarquons que

$$\lim_{n\to\infty} \frac{P_n'(z)}{nP_n(z)}$$

existe (pour plus de détails,(voir:[22,49]) . Notons cette limite par $M(z)$. Par conséquent

$$\lim_{n\to\infty} \frac{L_n(z)}{P_n(z)} = (z-\alpha) \lim_{n\to\infty} \frac{P_n'(z)}{nP_n(z)}$$

implique

$$1 = \lim_{n\to\infty} \frac{\frac{L_{n+1}(z)}{P_{n+1}(z)}}{\frac{L_n(z)}{P_n(z)}} = \lim_{n\to\infty} \frac{L_{n+1}(z)}{P_{n+1}(z)} \frac{P_n(z)}{L_n(z)} = \lim_{n\to\infty} \frac{L_{n+1}(z)}{L_n(z)} \frac{P_n(z)}{P_{n+1}(z)} \quad (3.5)$$

L'hypothèse $\mu' > 0$ p.p. sur le cercle implique (pour plus de détails,(voir:[22,49,51,40,47])

$$\lim_{n\to\infty} \frac{L_{n+1}(z)}{L_n(z)} = z \ , \qquad (3.6)$$

uniformément sur les compacts de $|z| > 1$.Les deux assertions (3.14) et (3.15) impliquent:

$$\lim_{n\to\infty} \frac{P_n(z)}{P_{n,+1}(z)} = \frac{1}{z}$$

On aura donc

$$\lim_{n\to\infty} \frac{L_{n+1}(z)}{L_n(z)} = \lim_{n\to\infty} \frac{P_{n+1}(z)}{P_n(z)} = z,$$

uniformément sur les compacts de $|z| > 1$.Ecrivons,

$$\frac{(n+1)\, L_n(z)}{(z-\xi)\, P_n(z)} = \frac{1}{z-\xi} + \frac{P_n'(z)}{P_n(z)} = \frac{1}{z-\xi} + \frac{d}{dz}(\log P_n(z))$$

La dérivée logarithmique et son interprétation dépend de l'ensemble des zéros du polynôme P_n.Remarquons que

$$\frac{((z-\xi)\, P_n(z))'}{(z-\xi)\, P_n(z)} = \frac{d}{dz}\left[\log (z-\alpha)\, P_n(z)\right] \qquad (3.7)$$

et

$$((z-\alpha)\, P_n(z))' = P_n(z) + (z-\alpha) P_n'(z)$$

Donc (3.16) et (1.29) impliquent

$$\frac{d}{dz}\left[\log (z-\alpha)\, P_n(z)\right] = (n+1)\, \frac{L_n(z)}{(z-\alpha)\, P_n(z)}$$

L'intégration des deux membres de cette égalité de z_1 à z donne:

$$\frac{(z-\alpha)\, P_n(z)}{(z_1-\alpha)\, P_n(z_1)} = \exp\left((n+1)\int_{z_1}^{z} \frac{L_n(t)}{(t-\alpha)\, P_n(t)} dt\right).$$

ce qui donne

$$P_n(z) = \frac{z_1-\alpha}{z-\alpha} P_n(z_1) \exp\left((n+1)\int_{z_1}^{z} \frac{L_n(t)}{(t-\alpha)\, P_n(t)} dt\right)$$

Par conséquent

$$\frac{P_{n+1}(z)}{P_{n,}(z)} = \tag{3.8}$$

$$\frac{P_{n+1}(z_1)}{P_n(z_1)} \exp\left[(n+2)\int_{z_1}^{z} \frac{L_{n+1}(t)}{(t-\alpha)P_{n+1}(t)}dt - (n+1)\int_{z_1}^{z} \frac{L_n(t)}{(t-\alpha)P_n(t)}dt\right]$$

ou encore

$$\frac{P_{n+1}(z)}{P_{n,}(z)}\left(\frac{P_{n+1}(z_1)}{P_n(z_1)}\right)^{-1} = \exp\left[(n+2)\int_{z_1}^{z} \frac{L_{n+1}(t)}{(t-\alpha)P_{n+1}(t)}dt - (n+1)\int_{z_1}^{z} \frac{L_n(t)}{(t-\alpha)P_n(t)}dt\right]$$

D'autre part, on a

$$\lim_{n\to\infty}\int_{z_1}^{z} \frac{L_n(t)}{(t-\alpha)P_n(t)}dt = \lim_{n\to\infty}\int_{z_1}^{z} \frac{L_{n+1}(t)}{(t-\alpha)P_{n+1}(t)}dt.$$

(3.18) et (3.19) impliquent

$$\lim_{n\to\infty}\frac{P_{n+1}(z)}{P_{n,}(z)}\left(\frac{P_{n+1}(z_1)}{P_n(z_1)}\right)^{-1} = \exp\left(\lim_{n\to\infty}(n+2-n-1)\int_{z_1}^{z} \frac{L_n(t)}{(t-\alpha)P_n(t)}dt\right)$$

Notons

$$\lim_{n\to\infty}\frac{P_n'(t)}{nP_{n,}(t)}dt = M(t)$$

et

$$\Lambda(z) = \int_{z_1}^{z} M(t)dt$$

Ceci donnent

$$\lim_{n\to\infty}\frac{P_{n+1}(z)}{P_{n,}(z)}\left(\frac{P_{n+1}(z_1)}{P_n(z_1)}\right)^{-1} = e^{\Lambda(z)}$$

avec

$$e^{\Lambda(z)} = \frac{z}{z_1}$$

On a donc

$$\lim_{n\to\infty}\frac{P_{n+1}(z)}{zP_n(z)} = \lim_{n\to\infty}\frac{L_{n+1}(z)}{zL_n(z)}$$

Ce qui achève la démonstration. ■

Théorème 3.2.**(voir:[26,27,28])** *Sous les mêmes hypothèses et notations du thérème 3.1, Notons par* $\{Q_n(z)\}_{n=0,1,2....}$ *la suite des polynômes polaires unitaires de seconde ordre associés à* μ. *dont le coefficient de* z^n *est égal à* $+1$. Alors*on a*

$$Lim_{n\longrightarrow\infty}\frac{Q_n'(z)}{nQ_n(z)} = \frac{1}{z} \tag{3.9}$$

uniformément sur les compacts de $\{|z|>1\}$ *.et*

$$\lim_{n\to\infty}\frac{Q_{n+1}(z)}{zQ_n(z)}=1 \tag{3.10}$$

encore

$$Lim_{n\longrightarrow\infty}\frac{Q_n(z)}{P_n(z)}=\frac{z}{z-\alpha} \tag{3.11}$$

uniformément sur les compacts de $\{z:|z|>1\}$ *.*

Preuve : Rappelons que par définition les polynômes polaires $Q_n(z)$ sont solutions de l'équation différentielle suivante:

$$\left(\log\left((z-\alpha)^2 Q_n(z)\right)'\right)'\left(\log\left((z-\alpha)^2 Q_n(z)\right)\right)'=\frac{(n+1)(n+2)}{(z-\alpha)^2}\frac{L_n(z)}{Q_n(z)}.$$

D'aprés (3.3) et (3.4), on remarquant que la dérivée logarithmique et son interprétation dépend de l'ensemble des zéros du polynôme Q_n, .on a

$$\frac{d}{dz}\log(z-\alpha)^2 Q_n(z)=(n+2)\frac{P_n(z)}{(z-\alpha)Q_n(z)} \tag{3.12}$$

L'intégration des deux membres de cette égalité de z_1 à z donne:

$$(z-\alpha)^2 Q_n(z)=(z_1-\alpha)^2 Q_n(z_1)\exp\left((n+2)\int_{z_1}^{z}\frac{P_n(t)}{(t-\alpha)Q_n(t)}dt\right)$$

Par conséquent

$$Q_n(z)=\frac{(z_1-\alpha)^2}{(z-\alpha)^2}Q_n(z_1)\exp\left((n+2)\int_{z_1}^{z}\frac{P_n(t)}{(t-\alpha)Q_n(t)}dt\right)$$

implique

$$Lim_{n\longrightarrow\infty}\frac{Q_{n+1}(z)}{Q_n(z)}\left(\frac{Q_{n+1}(z_1)}{Q_n(z_1)}\right)^{-1}=\exp\left(Lim_{n\longrightarrow\infty}\int_{z_1}^{z}\frac{P_n(t)}{(t-\alpha)Q_n(t)}dt\right)$$

Il vient donc,

$$Lim_{n\longrightarrow\infty}\frac{Q_{n+1}(z)}{Q_n(z)}\left(\frac{Q_{n+1}(z_1)}{Q_n(z_1)}\right)^{-1}=\exp\int_{z_1}^{z}F(t)dt$$

où

$$F(z)=Lim_{n\longrightarrow\infty}\frac{P_n(z)}{(z-\alpha)Q_n(z)}$$

D'aprés (1.3) et (1.4)

$$(z-\alpha)^2 Q_n'(z) + 2(z-\alpha) Q_n(z) = (n+2)(z-\alpha) P_n(z)$$

par conséquent,

$$F(z) = Lim_{n\longrightarrow\infty} \frac{P_n(z)}{(z-\alpha) Q_n(z)} = Lim_{n\longrightarrow\infty} \frac{Q_n'(z)}{nQ_n(z)}$$

On a donc

$$\lim_{n\to\infty} \frac{P_{n+1}(z)}{zP_n(z)} = \lim_{n\to\infty} \frac{Q_{n+1}(z)}{zQ_n(z)}$$

Ce qui achève la démonstration..

L'hypothèse $\mu' > 0$ p.p. sur le cercle implique (pour plus de détails,**(voir:[22,49,51,40,47]**)

$$\lim_{n\to\infty} \frac{Q_{n+1}(z)}{Q_n(z)} = z$$

uniformément sur les compacts de $|z| > 1$,et

$$\exp \int_{z_1}^{z} F(t)dt = \frac{z_1}{z}$$

Dérivons le deux membre de cettte égalié, on obtient

$$F(z) = Lim_{n\longrightarrow\infty} \frac{Q_n'(z)}{nQ_n(z)} = \frac{1}{z}$$

uniformément sur les compacts de $|z| > 1$,

3.2 Comportement asymptotique des polynômes réciproques des polynômes polaires primitives des POCU

Que se passe t'il pour les polynômes réciproques. C'est l'objet du théorème suivant

Théorème 3.3.**(voir:[26,27,28])** *Sous les mêmes hypothèses et notations du thérème 3.1, on a*

$$\lim_{n\to\infty} \frac{L_n^*(z)}{P_n^*(z)} = (1-\overline{\alpha}z) \lim_{n\to\infty} \frac{P_n^{*\prime}(z)}{nP_n^*(z)} \tag{3.13}$$

et

$$\lim_{n\to\infty} \frac{P_{n+1}^*(z)}{P_n^*(z)} = 1 \tag{3.14}$$

uniformément sur les compacts de $\{|z| < 1\}$.

Preuve: Il est utile de voir que

$$(n+1) L_n^* (z) = (1 - \overline{\alpha} z) P_n'^* (z) + P_{n,}^* (z)$$

et

$$(n+1) L_n^* (z) = (1 + n (1 - \overline{\alpha} z)) P_n^* (z) - z (1 - \overline{\alpha} z) P_n^{*\prime} (z)$$

autrement dit

$$\lim_{n\to\infty} \frac{L_n^* (z)}{L_{n-1}^* (z)} = \lim_{n\to\infty} \frac{n P_n^* (z)}{(n-1) P_{n-1}^* (z)} \frac{(1 - \overline{\alpha} z) - z (1 - \overline{\alpha} z) \frac{P_n^{*\prime}(z)}{n P_n^*(z)}}{(1 - \overline{\alpha} z) - z (1 - \overline{\alpha} z) \frac{P_{n-1,\xi}^{*\prime}(z)}{(n-1) P_{n-1}^*(z)}}$$

On combine avec (3.2), on obtient

$$\lim_{n\to\infty} \frac{P_n^* (z)}{P_{n-1}^* (z)} = 1$$

uniformément sur les compacts de $\{|z| < 1\}$. Ce qui achève la démonstration de l'assertion (3.13). Maintenant,observons que

$$\lim_{n\to\infty} \frac{L_n^* (z)}{P_n^* (z)} = (1 - \overline{\alpha} z) \lim_{n\to\infty} \frac{P_n^{*\prime} (z)}{n P_n^* (z)}$$

uniformément sur les compacts de$\{|z| < 1\}$. D'où l'assertion (3.14). ■

3.3 Comportement asymptotique des polynômes dérivées des polynômes polaires primitives des POCU

Théorème 3.4 (voir:[26,27,28])*Sous les mêmes hypothèses et notations du thérème 3.2., on a*

$$\lim_{n\to\infty} \frac{L_n' (z)}{P_n' (z)} = \frac{z - \alpha}{z} \tag{3.15}$$

et

$$\lim_{n\to\infty} \frac{P_{n+1}' (z)}{z P_n' (z)} = \lim_{n\to\infty} \frac{Q_{n+1}' (z)}{z Q_n' (z)} = 1 \tag{3.16}$$

uniformément sur les compacts de $\{z, |z| \geq 1\}$.

Preuve: Dérivons les deux membres de (1.3), on obtient

$$(n+1) L_n' (z) = (z - \alpha) P_n' (z) + P_n (z)$$

d'où

$$\lim_{n\to\infty} \frac{L_n' (z)}{P_n' (z)} = (z - \alpha) \lim_{n\to\infty} \frac{P_n' (z)}{n P_n (z)}$$

On obtient (3.15) en utilisant (3.3).D'où la conclusion de (3.15) et (3.16).

$$(z-\alpha)\,Q_n'(z)+Q_n(z)=(n+2)\,(z-\alpha)\,P_n(z)$$

implique

$$\lim_{n\to\infty}\frac{Q_n'(z)}{nQ_n(z)}=\lim_{n\to\infty}\frac{P_n'(z)}{nP_n(z)}=\frac{1}{z}$$

aussi

$$\lim_{n\to\infty}\frac{Q_n(z)}{P_n(z)}\text{ existe}$$

implque

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{Q_n(z)}{P_n(z)}\right)'=\lim_{n\to\infty}\frac{Q_n(z)}{P_n(z)}\left(\frac{Q_n'(z)}{nQ_n(z)}-\frac{P_n'(z)}{nP_n(z)}\right)=0$$

En effet

$$\left(\frac{Q}{P}\right)'=\frac{Q}{P}\left(\frac{Q'}{Q}-\frac{P'}{P}\right)$$

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Title: **Variational Analysis of a Dynamic Electroviscoelastic Problem with Friction**

Abstract: A dynamic contact problem is studied. The material behavior is modelled with piezoelectric effects for electro-visco-elastic constitutive law. The body may come into contact with a rigide obstacle. Contact is described with the Signorini condition, a version of Coulomb's law of dry friction, and a regularized electrical conductivity condition.We derive a variational formulation of the problem, then, under a smallness assumption on the coefficient of friction, we prove an existence and uniqueness result of a weak solution for the model. The proof is based on arguments of evolutionary variational inequalities and fixed points of operators.

The global existence of small data solutions of certain viscoelastic evolution problems

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Abstract. We establish some new results concerning the initial value problem first order on the whole space $\mathbb{R}^n$ $(n \geq 1)$, the decay structure of which is of regularity-loss property. By using Fourier transform and Laplace transform, we obtain the fundamental solutions and thus the solution to the corresponding linear problem. Appealing to the point-wise estimate in the Fourier space of solutions to the linear problem, we get estimates and properties of solution operator, by exploiting which decay estimates of solutions to the linear problem are obtained. Also by introducing a set of time-weighted Sobolev spaces and using the contraction mapping theorem, we obtain the global in-time existence and the optimal decay estimates of solutions to the semi-linear problem under smallness assumption on the initial data.

Approximate method for oxygen diffusion and absorption in sick cell

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Abstract

we consider the oxygen diffusion problem where the injection of oxygen into a sick cell and diffusion of the injected oxygen inside the cell. The problem mathematically formulated through two different steps. At the first stage, the stable case having no oxygen transition in the isolated cell is searched while at the second stage the moving boundary of oxygen absorbed by the tissues in the cell is searched. In this study, trace of moving boundary of the oxygen diffusion problem is determined using constrained integral method, the profile of moving boundary is determined by third order polynomial.

Keywords : Oxygen diffusion; Constrained integral method; Moving boundary problem; Stefan problem.
AMS Subject Classification : 35R35; 80A22; 65M06; 65N06.

1 Problem

The moving boundary problem arising in biomechanical diffusion theory which is formulated in Seval Çatal [4](2003). This type of problem was studied by Crank and Several authors [1], [2]. We see that the analytical solution is difficult to obtain and the moving boundary is an essential peculiarity of this problem. The oxygen diffusion in sick cell is generally presented in two stage. First oxygen is allowed to diffuse into a sick cell.The second stage is that of tracing the movement of the boundary and determining the distribution of the oxygen in the cell. We express in non-dimensional form, the problem is giving by (Seval Çatal 145 (2003) $361-369$)[4] is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 1 \quad 0 \leq x \leq s(t) \tag{1.1}$$

the boundary condition

$$\frac{\partial u}{\partial x} = 0 \quad \text{at the sealed surface} \quad x = 0 \quad t \geq 0 \tag{1.2}$$

$$u = \frac{\partial u}{\partial x} = 0 \quad \text{at the moving boundary} \quad x = s(t) \quad t \geq 0 \tag{1.3}$$

and the initial conditions at $t = 0$ are

$$u = \frac{1}{2}(1 - x)^2 \quad 0 \leq x \leq 1 \tag{1.4}$$

with

$$u = 0 \quad , \quad x = s(t) = 1 \quad \text{and} \quad t \geq 0 \tag{1.5}$$

We note that $u(x,t)$ is the concentration of oxygen free to diffuse at a point x, at time t and the location of the moving boundary is $s(t)$.

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Galerkin method for the higher dimension Boussinesq equation non linear with integral condition

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Abstract-This paper deals with the solvability of a higher dimension mixed non local problem for a Boussinesq equation non linear. Galerkin's method was the main used tool for proving the solvability of the given non local problem.
Keywords: Boussinesq equation, non local condition, Galerkin's Method
2000 Mathematics Subject Classification: 35L20, 58J45.

1 Introduction

By applying mathematical modeling to various phenomena of physics, biology and ecology there often arise problems with non-classical boundary conditions, which connect the values of the unknown function on the boundary and inside of the given domain. Some times the physical phenomena are modeled by non classical boundary value problems which involve a boundary condition as an integral condition over the spatial domain of a function of the desired solution. The nonlocal boundary condition arises mainly when the data on the boundary cannot be measured directly, but their average values are known. In the very recent years, nonlocal problems, particularly those with integral constraints have received great attention. The physical significance of nonlocal conditions such as a mean, total mass, moments, etc, has served as a fundamental cause for the considerably increasing interest to this kind of boundary value problems. Nonlocal problems are generally encountered in chemical engineering, heat transmission, plasma physics, heat transmission, thermoelsticity and underground water flow. See in this regard the papers by Ewing and Lin [3], Choi and Chan [2]. As a special application see Bouziani [1], where the author has considered a nonlocal problem which is proposed in the mathematical modeling of technologic process of external elimination of gas, practices in the refining of impurities of Silicon lamina.

In section 1, we state the problem, define some spaces and give a relevant definition of weak solution. Section 2 is devoted to the study of existence of the weak solution of the posed problem by applying Galerkin's method.

In this paper, we are concerned with the following nonlocal mixed boundary value problem for the $n-$dimensional Boussinesq equation non linear in a cylinder $Q_T = \Omega \times (0, T)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

$$\begin{cases} u_{tt} - \alpha^2 \Delta u - \beta^2 \Delta u_{tt} = |u|^{p-2} u, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \\ \frac{\partial u}{\partial \eta} = \int_0^t \int_\Omega u(\xi,\tau) d\xi d\tau, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $p > 2$, $\varphi(x)$ and $\psi(x)$ are given functions and $\frac{\partial u}{\partial \eta}$ designates the normal derivative.
Now let $V(Q_T)$ and $W(Q_T)$ be the set spaces defined respectively by:

$$V(Q_T) = \left\{u \in W_2^1(Q_T) : \nabla u_t \in L^2(Q_T), u \in L^p(Q_T),\ u_t \in L^p(Q_T)\right\},$$

and

$$W(Q_T) = \{u \in V(Q_T) :\ v(x,T) = 0\}. \tag{2}$$

Consider the equation

$$(u_{tt}, v)_{L^2(Q_T)} - \alpha^2 (\Delta u, v)_{L^2(Q_T)} - \beta^2 (\Delta u_{tt}, v)_{L^2(Q_T)} = \left(|u|^{p-2} u, v\right)_{L^2(Q_T)}. \tag{3}$$

Evaluation of the inner products in (3) and use of boundary condition in (1) leads

$$\begin{aligned}
& -(u_t, v_t)_{L^2(Q_T)} + \alpha^2 (\nabla u, \nabla v)_{L^2(Q_T)} - \beta^2 (\nabla u_t, \nabla v_t)_{L^2(Q_T)} \\
= \; & \left(|u|^{p-2} u, v\right)_{L^2(Q_T)} - (\psi(x), v(x,0))_{L^2(\Omega)} + \alpha^2 \int_{\partial\Omega}\int_0^T v(x,t) \left(\int_0^t \int_\Omega u(\xi,\tau) d\xi\right) dt ds_x \\
& + \beta^2 \int_{\partial\Omega}\int_0^T v(x,t) \left(\int_\Omega u_t(\xi,t) d\xi\right) dt ds_x - \beta^2 \int_{\partial\Omega}\int_0^T v(x,t) \left(\int_\Omega u_t(\xi,0) d\xi\right) dt ds_x \\
& + \beta^2 (\nabla\psi(x), \nabla v(x,0))_{L^2(\Omega)},
\end{aligned} \tag{4}$$

$\forall v \in W(Q_T)$.
Definition 1.1. A function $u \in V(Q_T)$ is called a generalized solution of problem (1), if it satisfies equation (4) for each $v \in W(Q_T)$ and $u(x,0) = \varphi(x)$.

2 Solvability of the problem

We now give the main result on the existence of solution of problem (1) and prove it by using the Galerkin method.

Theorem 1 *If $\varphi(x) \in W_2^1(\Omega)$, $\psi(x) \in L^p(\Omega)$ and $\psi(x,t) \in W_2^1(\Omega)$, then there is at least one generalized solution in $V(Q_T)$ to problem* (1).

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ASYMPTOTIC STABILITY OF A PROBLEM WITH KELVIN-VOIGT TERM AND BALAKRISHNAN-TAYLOR DAMPING.

TOUALBIA SARRA

ABSTRACT. The purpose of this work is to study the energy Decay of solutions for a nonlinear equation of the kelving voigt type with balakrishnan taylor damping and acoustic boundary in a bounded domain in $\mathbb{R}^n$.

1. INTRODUCTION

we investigate for the following problem

$$\left|u'\right|^m u'' = \left(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma \int_\Omega \nabla u.\nabla u' dx\right)\Delta u + 2\lambda\Delta u' \quad \text{in } \Omega\times\mathbb{R}^+, \quad (1)$$

$$u = 0 \quad \text{in } \Gamma_0\times\mathbb{R}^+, \quad (2)$$

$$\left(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma \int_\Omega \nabla u.\nabla u' dx\right)\frac{\partial u}{\partial v} + 2\lambda\frac{\partial u'}{\partial v} = y' \quad \text{in } \Gamma_1\times\mathbb{R}^+, \quad (3)$$

$$u' + p(x)y' + q(x)y = 0 \quad \text{in } \Gamma_1\times\mathbb{R}^+, \quad (4)$$

$$u(0) = v_0,\ u'(0) = u_1 \quad \text{in } \Omega, \quad (5)$$

$$y(0) = y_0 \quad \text{in } \Gamma_1. \quad (6)$$

where Ω a bounded, connected set in $\mathbb{R}^n (n \geq 1)$ having a smooth boundary $\Gamma = \partial\Omega$ consisting of two parts Γ_0 and Γ_1 such that $\overline{\Gamma_0}\cup\overline{\Gamma_1} = \Gamma$. Primes denote the time derivative, Δ the Laplacian in $\mathbb{R}^n$ taken in space variables, v the unit

normal of Γ pointing towards exterior of Ω and $\mathbb{R}^+ = (0,\infty)$ The parameters $\lambda > 0$ is a small internal material damping coefficient, $a > 0$, $b > 0$, $\sigma > 0$. are constant real numbers p and q are functions satisfying $p(x) > 0$, $q(x) > 0$ fo rall $x\in\Gamma_1$.

Theorem 1.1. *For the initial data* $(u_0, u_1, y_0) \in \left(V\cap H^2(\Omega)\right)\times V\times L^2(\Gamma_1)$ *,there exists a unique pair of functions* $(u; y)$*, which is a solution to the problem (1)-(6) in the class*

$$\begin{aligned} u \in L^\infty\left(0,T; V\times H^2(\Omega)\right),\ u_t \in L^\infty(0,T;V) \\ u_{tt}\in L^\infty\left(0,T;\times L^2(\Omega)\right), y, y_t \in L^\infty\left(0,\infty; L^2(\Gamma_1)\right). \end{aligned} \quad (7)$$

We defined the functional energyof system (1-6) by

$$E(t) = \frac{1}{m+2}\|u_t(x,t)\|_{m+2}^{m+2} + \frac{1}{2}\left(a^2 + \frac{b}{2}\|\nabla u(x,t)\|_2^2\right)\|\nabla u(x,t)\|_2^2 + \frac{1}{2}\int_{\Gamma_1} q(x)(y(t))^2 d\Gamma. \quad (8)$$

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Lemma 1.2. *For every solution $u(x;t)$ of the system (1-6), the time derivative of the functional*

$$\Psi(t) = \frac{1}{m+1}\int_{\Omega} |u_t|^m u_t u dx + \lambda \int_{\Omega} |\nabla u|^2 dx + \frac{\sigma}{4}\left(\int_{\Omega} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\Gamma_1} p(x) y^2 d\Gamma + \int_{\Gamma_1} u y d\Gamma, \tag{9}$$

satisfies

$$\Psi^{'}(t) \leq \frac{1}{m+1}\|u_t\|_{m+2}^{m+2} - 2\int_{\Gamma_1} u y_t d\Gamma - \int_{\Gamma_1} q(x) y^2 d\Gamma - a^2 \|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \frac{\sigma}{4}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2. \tag{10}$$

Il existe deux constantes positives M_1, M_2, telles que

$$(1-\varepsilon M_1) E(t) \leq V(t) \leq (1+\varepsilon M_2) E(t), \ \forall t \geq 0, \tag{11}$$

où

$$0 < \varepsilon < \frac{1}{M_1}.$$

Theorem 1.3. *If $u(x;t)$ is a regular solution of the system (1-6) with initial values $(u_0, u_1, y_0) \in \left(V \cap H^2(\Omega)\right) \times V \times L^2(\Gamma_1)$ then the energy $E(t)$ of the system de ned by (1-6) satis es*

$$E(t) < Me^{-\mu t}E(0), \ \ pour \ t \in (0, \infty), \tag{12}$$

for some real constants $M > 1$ and $\mu > 0$.

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2-Orthogonal Polynomials and Darboux Transforms

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Abstract

In this work we present a new interpretation of Darboux transforms in the context of 2-orthogonal polynomials and find conditions in order for any Darboux transform to yield a new set of 2- orthogonal polynomials. We also introduce the LU and UL factorizations of the monic Jacobi matrix associated with a quasi-definite linear functional defined on the linear space of polynomials with real coefficients.

In 2004, M. I. Bueno, F. Marcellán [2], introduced the LU and UL factorizations of a tridiagonal matrix J, as well as the transformation of Darboux and the Darboux transformation without parameters. They also show how to find the tridiagonal matrix J_1^n associated with the linear functional $\Gamma_1 = x\ \Gamma$ in terms of the matrix J by the application of the Darboux transformation without parameters.

The main purpose of this work is to present a new interpretation of Darboux transforms in the context of $2-$ orthogonal polynomials.

Keywords: 2 orthogonal polynomials, linear functional, Darboux transformation

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The method (HPM) for solving some problems of heat-like equations with non local boundary conditions.

Abstract: In this work initial boundary value problems are presented. The homotopy perturbation method (HPM) is used for solving some problems of heat-like equations with non local boundary conditions. The obtained results are highly accurate. This method provides continuous solutions in contrast to other numerical methods, like finite difference, finite elements, spectral methods, ect. It is found that this method it is a powerful tool and can be applied to a large class of linear and non linear problems in different fields of science and engineering.

Key Words: Homotopy perturbation method (HPM), partial differential equations, initial boundary value problems, non local boundary conditions.

Introduction

La modélisation du plusieurs systèmes physiques conduit à des équations fonctionnelles, dans différent champs physiques et dans l'ingénierie. Dans les deux dernières décennies beaucoup de chercheurs se sont intéressés à des méthodes analytiques Pour résoudre les équations aux dérivées partielles avec des conditions aux limites non locales et parmi ces méthodes, la méthode de perturbation d'homotopie. Cette méthode a été initié et introduite la première fois par le mathématicien chinois J. He en [4-5] qui l'a appliqué pour résoudre l'équation d'ondes et d'autres problèmes aux limites avec des conditions initiales. Après lui d'autres chercheurs comme [6], [1-3] et autres ont appliqué cette méthode pour résoudre l'équation de la chaleur et l'équation de la chaleur-semblable (Heat-Like équation). Dans cet article on propose un problème de la chaleur-semblable présenté sous la forme générale

(1) $$u_t - G(x,t,u,u_x,u_{xx}) = 0, a < x < b, 0 < t < T$$

A condition initiale

(2) $$u(x,0) = f(x),\ \ a < x < b$$

Et les conditions aux limites non locales

(3) $$u(a,t) = \int_a^b \varphi(x,t)u(x,t)dx + g_0(t)\,, 0 < t < T$$

(4) $$u(b,t) = \int_a^b \psi(x,t)u(x,t)\,dx + g_1(t), 0 < t < T$$

Où $f, \varphi, \psi, g_0\ et\ g_1$ sont des fonctions connues et continues. Ensuite on résout ce problème en utilisant la méthode de perturbation d'homotopie. Toutes les solutions obtenues agréent avec les solutions exactes.

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Exponential decay for a nonlinear axially moving viscoelastic string

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Abstract

The stabilization of a nonlinear axially moving viscoelastic string is the topic of this paper. Next, we are showing Under reasonable conditions on the initial results, by using the prospective well process, certain solutions exist globally. We then demonstrate that the damping provided by the viscoelastic term is sufficient to ensure an exponential decay.
Key words: moving string,Arbitrary decay ,multiplier method, Asymptotic behavior, Stability. Stability.

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APPLICATION OF METAHEURISTIC ALGORITHM IN FINDING THE CURRENT IVP's EXPRESSION OF AN ELECTRIC CIRCUIT

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ABSTRACT

The differential equations that describe many attractive natural phenomena are one of the most motivating fields of mathematics. Solving Initial Value Problems (IVPs) in ordinary differential equations (ODEs) via fundamental mathematical methods gives in general insuitable results especially face to difficult problems. Hence the solution of this lacks is found by using a meta-heuristic algorithms. In this paper we propose by means of the Flower Pollination Algorithm (FPA) (Xin-She Yang, 2013) how to solve the IVPs arising from a circuit consisting of a resistor and a capacitor in both constant voltage and variable voltage cases. The conducted comparison between the exact solution and the algorithm outcomes in the investigated examples showed that the FPA yields satisfactorily precise approximation of the solutions.

Keywords: ODEs; IVPs; FPA; Series RC circuit; Optimisation Problems.

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1. INTRODUCTION

Optimization is a process that locates a best, or optimal values of the variables that minimize or maximize the objective function while satisfying the constraints, it arises in various disciplines. Engineering problems under growing dimensions, moment complexity, variables, and space complication are becoming more and more difficult and hard to optimize; consequently optimization algorithms are then used to overcome this situation but traditional optimization techniques, including many heuristic approaches still insufficient.

To cope up with such situation, many researchers focus on nature by creating a large collection of meta- heuristic algorithms which is characterized by its convergence speed and augmentation searched variables number. These methods generate a simpler procedure to solve an optimization problem to find good solutions with less computational effort than simple algorithms or traditional heuristics [3] [4].

Meta- heuristic algorithms take dissimilar forms according to the inspired process of the natural systems like Genetic algorithm [8] [10], Ant colony optimization algorithm [5], Bee algorithm [15] [6], Particle Swarm Optimization [13], Bat algorithm [24]...etc. All these algorithms have several advantages illustrated via a wide range of applications.

The Flower Pollination Algorithm (FPA) [23] is a recent bio-inspired optimization algorithm that takes off the real life processes of the fertilization (pollination) process of flowers. In FPA, abiotic pollination is considered for local pollination while biotic pollination is considered for the global pollination between the flower plants. The algorithm maintains a balance between local and global pollination. It tooks an interesting place between the more recent nature inspired algorithms kept by its nice performance against several classical meta-heuristic algorithms. This is behind the vast utilizations of FPA in various domains such as chemical engineering, civil engineering, communication engineering, medical field, computer science…etc. FPA was hybridized with other nature inspired meta-heuristic algorithms in order to overcome its limitations and to benefit from their strength e.g. PSO [25], frog leaping local search [14] and simulated annealing [1], Bat algorithm [26] …etc.

In electronics and electric engineering a first order RC circuit (RC filter or RC network) [12]. It is an electric circuit composed of resistors and capacitors, either in series driven by a voltage source or in parallel driven by a current source [11]. The importance of this study is to consider the ODEs arising from a series RC circuit in both constant voltage and variable voltage cases as an IVPs then they are formulated as an optimization problem [16], when the FPA [23] is used as a tool to find numerical solutions for this problem.

The remainder paper is organized as follows. The formulation of the problem is revealed in section 2; section 3 provides basics on FPA and its main steps for finding an approximate solution of IVP. The Section 4 gives essential formulae with brief explication of series RC circuit ODEs. The Section 5 exposes examples of series RC circuit IVPs to show how the FPA can lead to a satisfactory result for solving IVP. The comments and conclusion are made in section 6.

2. PROBLEM FORMULATION

A first order Initial Value Problem (IVP) is defined as a real function of two real finite variables *a* and *b* when we look to find a function y(x), continuous and differentiable for $x \square [a,b]$ such that $y'=f(x,y)$ from $y(a)=y_0$ for all $x \in [a,b]$ [9] for all real values of *y*. The equations (1) explain an IVP:

$$\begin{cases} y' = f(x,y) \\ y(a) = y_0 \end{cases} \quad (1)$$

This problem have unique solution if: *f* is continuous on $[a,b] \times \mathbb{R}$, and satisfies the Lipschitz condition; it exists a real constant $k>0$, as $|f(x,\theta_1)-f(x,\theta_2)| \leq k| \theta_1 - \theta_2 |$, for all $x \in [a,b]$ and all couple $(\theta_1,\theta_2) \in \mathbb{R} \times \mathbb{R}$.

Finding the optimal solutions numerically of an IVP is gotten with approximations: $y(x_0+h),\dots,y(x_0+nh)$ where $a=x_0$ and $h=(b-a)/n+1$. For more precision of the solution, we must use a very small step size *h* that includes a larger number of steps, thus more computing time which is not available in the useful numerical methods that may approximate solutions of IVP and perhaps yield useful information, often sufficing in the absence of exact, analytic solutions like Euler and Runge-Kutta methods [9].

2.1. Objective function

The main idea in the formulation of the objective function is to use the finite difference formula for the derivative and equation (1) we obtain,

$$\frac{y(x_j) - y(x_{j-1})}{h} \approx f\left(x_{j-1}, y\left(x_{j-1}\right)\right).$$

hence,

$$\frac{y_j - y_{j-1}}{h} \approx f(x_{j-1}, y_{j-1}).$$

Therefore, we have to consider the error formula:

$$[\frac{y_j - y_{j-1}}{h} - f(x_{j-1}, y_{j-1})]^{2}$$

The objective function, associated to Y= $(y_1, y_2, \dots, y_{10})$ will be:

$$F(y) = \sum_{i=1}^{d}[\frac{y_j - y_{j-1}}{h} - f(x_{j-1}, y_{j-1})]^{2} \tag{2}$$

2.2. Consistency

We are interested in the calculation of Y=$(y_1, y_2, \dots, y_{10})$ which minimizes the objective function in equation (2). We have from Taylor's formula order 1;

$$y_j = y_{j-1} + hy'_{j-1} + O(h^2), \qquad j = 1, \dots, d.$$

So,

$$\frac{y_j - y_{j-1}}{h} = y'_{j-1} + O(h)$$

If we subtract $f(x_{j-1}, y_{j-1})$ from both sides of last equation, we obtain

$$\frac{y_j - y_{j-1}}{h} - f\left(x_{j-1}, y_{j-1}\right) = y'_{j-1} - f\left(x_{j-1}, y_{j-1}\right) + O(h), \qquad j=1,\dots,d.$$

The last relation shows that the final value Y=$(y_1, y_2, \dots, y_{10})$ is an approximate solution of IVP, for small value of *h*.

3. FLOWER POLLINATION ALGORITHM (FPA)

3.1. Flower Pollination description

Pollination is very important. It leads to the creation of new seeds that grow into new plants. It begins in the flower. Flowering plants have several different parts that are important in pollination. Flowers have male parts called stamens that produce a sticky powder called pollen. Flowers also have a female part called the pistil. The top of the pistil is called the stigma, and is often sticky. Seeds are made at the base of the pistil, in the ovule. To be pollinated, pollen must be moved from a stamen to the stigma [22]. There are two types of pollination:

1. Self Pollination (Abiotic pollination): Only about 10% of plants fall in this category, it's the fertilization of one flower, when the pollen from a flower pollinates the same flower or flowers of the same plant; it does not require any pollinators. It occurs when a flower contains both the male and the female gametes, It is a process where the pollination happens without involvement of external agents [21] (Figure. 1)

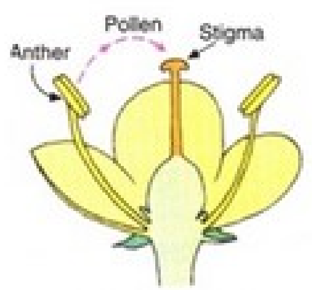


Fig. 1 Self pollination.

2. Cross Pollination (biotic pollination): Is typically associated when pollen from a plant's stamen is transferred to a different plant's stigma (of the same species), and such transfer is often linked with pollinators. Pollination occurs in several ways [19]:

People: They can transfer pollen from one flower to another, but most plants are pollinated without any help from people.

Animals: such as bees, butterflies, moths, flies pollinate plants by an accidental way when they are at the plant to get food. The pollinators can fly a long distance, thus they can consider as the global pollination [7]. In addition, bees and birds may behave as Lévy flight behavior [17] [23], with jump or fly distance steps obey a Lévy distribution. Furthermore, flower constancy can be used an increment step using the similarity or difference of two flowers [7] [20].

Wind and Diffusion in water: it picks up pollen from one plant and blows it onto another [19] (Figure. 2)

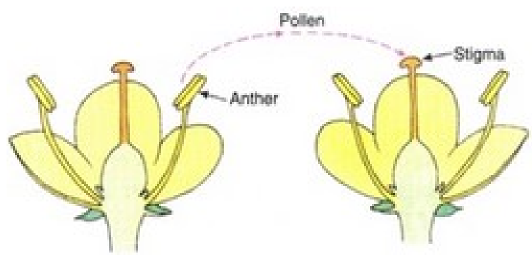


Fig. 2 Cross pollination.

3.2. Formulation of Flower Pollination Algorithm

The four rules given below are used to summarize the above characteristics of pollination process, flower constancy and pollinator behaviour [23].

1. Biotic and cross-pollination is considered as global pollination process and pollinators carrying pollen move in a way that confirms to Levy flights.
2. For local pollination, abiotic pollination and self-pollination are used.
3. Flower constancy can be considered as the reproduction probability is proportional to the similarity of two flowers involved.
4. Local pollination and global pollination is controlled by a switch probability $p \in [0,1]$. In principle, flower pollination process can happen at both local and global levels. But in reality, flowers in the neighborhood have higher chances of getting pollinated by pollen from local flowers than those which are far away. To simulate this feature, a proximity probability (Rule 4) can be commendably used to switch between intensive local pollination to common global pollination. To start with, a raw value of p=0.5 may be used as an initial value. A preliminary parametric study indicated that p=0.8 may work better for most applications

To formulate the updating formulas, these rules have to be changed into correct updating equations. The main steps of FPA or simply the flower algorithm are illustrated below:

Pseudo code of the proposed Flower Pollination Algorithm

Objective min or max f(x), $x=(x_1,x_2,..., x_d)$
Initialize a population of *n* flowers/pollen gametes with random solutions
Find the best solution g_* in the initial population
Define a switch probability *p∈[0,1]*
while *(t <MaxGeneration)*
  for *i=1:n* (all *n* flowers in the population)
    if *rand<p*,
      Draw a (*d*-dimensional) step vector *L* which obeys a Levy distribution
      Global pollination via $x_i^{t+1} = x_i^t + L(g_* - x_i^t)$
    else
    Draw from a uniform distribution in *[0,1]*
    Randomly choose *j* and *k* among all the solutions
    Do local pollination via $x_i^{t+1} = x_i^t + \epsilon(x_j^t - x_k^t)$
    end if
    Evaluate new solutions
    If new solutions are better, update them in the population
  end for
  Find the current best solution g_*
end while

4. CASE STUDY: SOLVING IVP FOR A SERIES RC CIRCUIT

The RC circuit (RC filter or RC network) is an electric circuit composed of resistors and capacitors driven by a voltage or current source, when the circuit is composed of one resistor and one capacitor then it's called a first order RC circuit which is the simplest type of RC circuit [11].

RC circuits have a several utilities, its may be used to filter a signal by blocking certain frequencies and passing others and charge transport behaviour in various complex systems described using models of many-element RC networks like the battery anodes and fuel cells [12]. The two most common RC filters are the high-pass filters and low-pass filters; band-pass filters and band-stop filters usually require RLC filters, though crude ones can be made with RC filters [2].

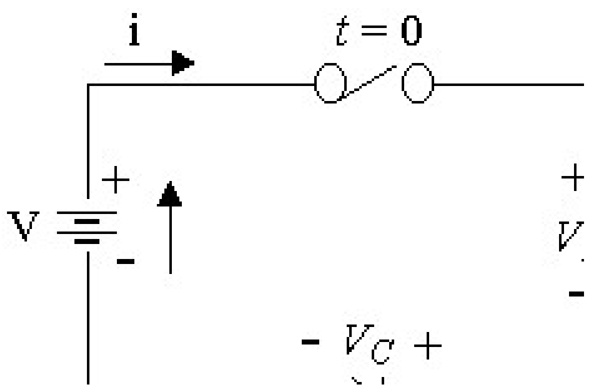


Fig. 3 The RC circuit diagram.

Case 1: Constant Voltage

The voltage across the resistor and capacitor are as follows : $V_R = Ri$ and $V_C = \frac{1}{C}\int idt$, Kirchhoff's voltage law says the total voltages must be zero. So applying this law to a series RC circuit results in the equation:

$$Ri + \frac{1}{C}\int idt = V$$

One way to solve this equation is to turn it into a differential equation, by differentiating throughout with respect to *t*:

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

Solving the equation gives us:

$$i = \frac{V}{R}e^{-t/RC} \quad (3)$$

The time constant in the case of an RC circuit is:

$$\tau = RC \quad (4)$$

Case 2: Variable Voltage and 2-mesh Circuits

We need to solve variable voltage cases in q, rather than in i, since we have an integral to deal with if we use i. So we will make the substitutions: $i = \frac{dq}{dt}$ and $q = \int idt$

So the equation in i involving an integral: $Ri + \frac{1}{C}\int idt = V$ becomes the differential equation in q:

$$R\frac{dq}{dt} + \frac{1}{C}q = V \tag{5}$$

5. NUMERICAL RESULTS

All through our experimental study, the FPA and its operators are coded in MATLAB (R 2013a) for the solution of IVP in the company of Euler scheme. The numerical results are exposed in graphical and tabular form. In order to give more justification to the investigated meta-heuristi algorithm comparaisons with the classical Euler method are made.

The problem treatment demanding, two types of parameters, the first are related to FPA and the second are connected to IVP. These parameters are described as follows:

1. **FPA related parameter**: In this work, the parameters adopted by the FPA in each problem are summarized in the following table:

Table 1. Parameters adopted by the FPA

Parameter	**Quantity**
Dimension of the search variables (d)	10
Total number of iterations (N)	2000
Population size (n)	20
Probability switch (p)	0.8

2. **IVP related parameter**: FPA is an optimization instrument. Then, the essential differential equation is converting into discretization form. The backward difference formula is used to convert differential equation into discretization form when the derivative term is replaced in the discretized form by a difference quotient for approximations.

   The interval of the IVP is equally partitioned into (n+ 1) equidistant subinterval with the length h= (b-a)/n+1. Where n = 9 is a number of interior nodes. The IVP related parameters are as follows:

   1. the number of interior nodes (n = 9).

2. The step size h = 0.5, h=0.02 for case1 and case2 respectively.

3. The initial condition in the first case is i=0 for t=0 and for the second case we assume that the charge on the capacitor is -0.05 C for t = 0. The interval between which the differential equation is solved is varying from case to case.

In the experimental study we introduce two IVP that arising from a RC circuit in both constant voltage and variable voltage cases. The objective function defined as

$$F(y_1, y_2, \dots, y_{10}) = \sum_{i=1}^{10} \left[\frac{y_j - y_{j-1}}{h} - f(x_{j-1}, y_{j-1})\right]^2 = \sum_{i=1}^{10} \left[\frac{y_j - y_{j-1}}{h} - y_{j-1}\right]^2 \quad (6)$$

Example 1 (Case 1: Constant Voltage) Finding the current in the RC circuit for t > 0 that has an emf of 100 V, a resistance R = 50 W, C = 0.02 F and no initial current by using the nature inspired algorithm FPA is done by means of equation (3) that gives $i = 2e^{-t}$ then by using equation (6) the numerical results of the comparaison between both FPA and Euler method results are sumarized in table (2). The time constant in this case is calculated via equation (4) gives τ=1 Seconds.

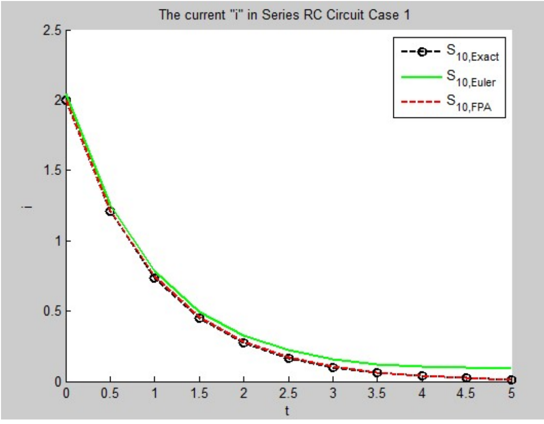
Table 2. Numerical results of the Case 1 example for d=10

i	t_i	**Exact Results**	**FPA Results**	**Euler Results**
0	0.0000	2.0000	2.0002	2.0402
1	0.5000	1.2131	1.2163	1.2569
2	1.0000	0.7358	0.7465	0.7845
3	1.5000	0.4463	0.4583	0.4964
4	2.0000	0.2707	0.2803	0.3253
5	2.5000	0.1642	0.1696	0.2223
6	3.0000	0.0996	0.1041	0.1598
7	3.5000	0.0604	0.0637	0.1228
8	4.0000	0.0366	0.0387	0.1084
9	4.5000	0.0222	0.0240	0.0978
10	5.0000	0.0135	0.0145	0.0912

The graphical representation of Table (1) results is visualized via Figure (4) that shows an exponential decay shape which means the current stops flowing as the capacitor becomes fully charged. A straightforward remark detection of difference between FPA performance and

Euler's performance is very clear; hence FPA is better than Euler because its results curve is very close to the exact results curve contrary to Euler method.

Fig. 4 The graphical representation of Example 1 results.

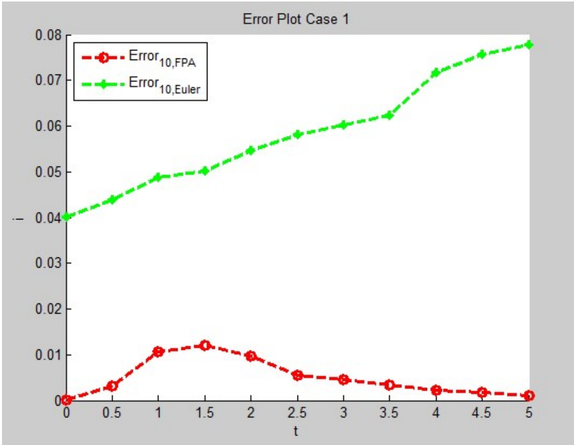


The absolute error between exact and FPA results and Euler method results are made in Table (3) as well as their graphical representations wich is gived through Figure (5). In both representations of the absolute error, FPA method provides a very minimal absolute error compared to Euler method.

Table 3. Absolute Error of Case 1 example for d=10

i	t_i	**FPA**	**Euler**
0	00000	0.0002	0.0402
1	0.5000	0.0032	0.0438
2	1.0000	0.0107	0.0487
3	1.5000	0.0120	0.0501
4	2.0000	0.0096	0.0546
5	2.5000	0.0054	0.0581
6	3.0000	0.0045	0.0602
7	3.5000	0.0033	0.0624
8	4.0000	0.0021	0.0718
9	4.5000	0.0018	0.0756
10	5.0000	0.0010	0.0777

Fig. 5 Absolute Error plot of Example 1 results.



Example 2 (Case 2: Variable Voltage and 2-mesh Circuits) Find the charge and the current for t > 0 in a series RC circuit where R = 10 W, C = 4 × 10-3 F and E = 85 cos 150t V. Assume that when the switch is closed at t = 0, the charge on the capacitor is -0.05 C. Since the voltage source is not constant, we cannot use the formulae in Equation (3), and from the formula of Equation (5) we have:

$$\frac{dq}{dt} + 25q = 8.5cos150t$$

Now, we can solve this differential equation in q using the linear ODE process so this gives us:

$$q(t) = 00092cos150t + 0.055sin150t - 0.059e^{-25t}$$

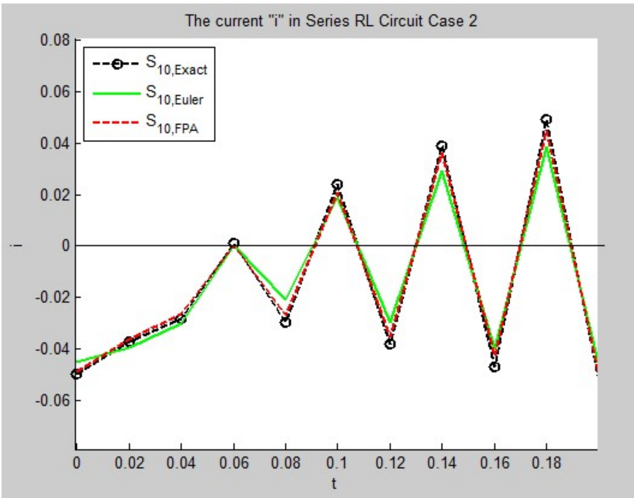
Table (4) recapitulates the FPA and Euler outcomes for case 2 example.

Table 4. Numerical results of the Case 2 example for d=10

i	t_i	Exact Results	FPA Results	Euler Results
0	00000	-0.0498	-0.0491	-0.0451
1	0.0200	-0.0371	-0.0363	-0.0395
2	0.0400	-0.0282	-0.0264	-0.0304
3	0.0600	0.0011	0.0002	0.0000
4	0.0800	-0.0297	-0.0267	-0.0207
5	0.1000	0.0239	0.0209	0.0189
6	0.1200	-0.0382	-0.0352	-0.0297
7	0.1400	0.0392	0.0362	0.0292
8	0.1600	-0.0470	-0.0429	-0.0399
9	0.1800	0.0493	0.0453	0.0388
10	0.2000	-0.0533	-0.0509	-0.0469

Figure (5) illustrates the graphical representations of the exact results of example 2 and simulation via FPA and Euler method, Hence the FPA graph is extremely close to the exact results graph than Euler's one. We note that the graph can be very smooth by augmenting the number of steps *h*.

Fig. 5 The graphical representation of Example 2 results.

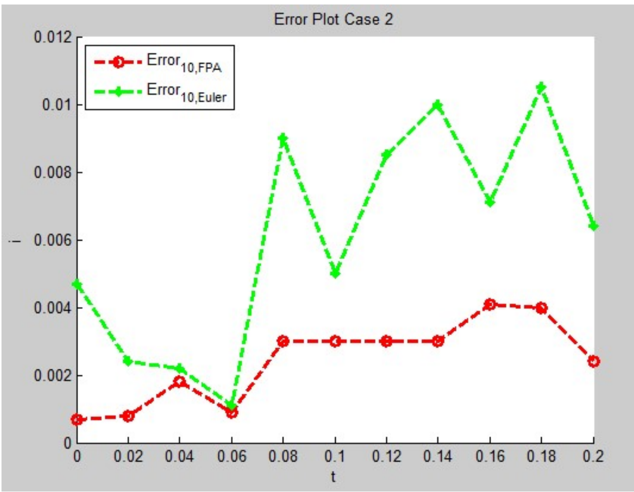


In order to validate the results of Table (4) and by means of Table (5) and Figure (6) that represent the absolte error study of these example in both tabular and grapgical forms we easily constate that the FPA gives better results than Euler since its possede a smallest error.

Table 5. Absolute Error of Case 2 example for d=10

i	t_i	FPA	Euler
0	00000	0.0007	0.0047
1	0.0200	0.0008	0.0024
2	0.0400	0.0018	0.0022
3	0.0600	0.0009	0.0011
4	0.0800	0.0030	0.0090
5	0.1000	0.0030	0.0050
6	0.1200	0.0030	0.0085
7	0.1400	0.0030	0.0100
8	0.1600	0.0041	0.0071
9	0.1800	0.0040	0.0105
10	0.2000	0.0024	0.0064

Fig. 7 Absolute Error plot of Example 2 results.



Discussions: The FPA have successfully developed to take off the characteristics of flower pollination. Our simulation results indicate that FPA is simple, reduces time, flexible and exponentially better to solve optimization IVP.

6. CONCLUSION

In this study, we applied the FPA to solve approximately the IVPs arises in electronic engineering field that are ODEs of the series RC circuit via a chosen examples in both voltage constant and voltage variable cases. After a comparison between the exact solutions and the algorithm outcomes with Euler method results; FPA conduct to a precise solutions with least error compared to the Euler’s one. That is another argument gived by the met-heuristics algorithms in demonstrating such good proprieties.

Behind the evaluations of various research papers, FPA was found as an algorithm having fabulous aptitude to solve a variety of optimization problems. As a futre reaserch, there are profound studies on FPA that will give hopeful results such as the use of more diverse parameters, more extensive comparison studies with more open sort of algorithms; for this reason these comparisons will enhance the qualities and back up the limitations of all the algorithms. Also, FPA should be looked into in several applications of engineering and industrial optimization problems.

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APPLICATION ON FRACTIONAL QUANTUM MECHANICS

BOUZENNA FATMA EL-GENBAZIA & MEFTAH MOHAMMED TAYEB

ABSTRACT. In this work, we solve the space fractional Schrödinger equation based on non singular Caputo-Fabrizio derivative definition for 1D linear potential. To reach this goal, we first work out the fractional differential equation defined in terms of Caputo-Fabrizio derivative. Hence, the wave functions of fractional Schrödinger equations are derived.

fractional Schrödinger equation; linear potential; fractional differential equation; Caputo-Fabrizio derivative

1. INTRODUCTION

The theory of the fractional derivation and integration has long been considered as a branch of mathematics without any real or practical explanation. Over the last three decades, considerable attention has been paid to the fractional calculus by applying these concepts in different fields of physics and engineering [1]. Recently, the fractional notion enters the world of quantum mechanics for the purpose of generalization without any contradiction with the postulates of standard quantum mechanics. The possibilities of this generalization was proven by Laskin [2], who developed a new fractional quantum mechanics, and was realized using the Feynman's path-integral approach [3]. In the standard quantum mechanics, Feynman's approach is based on the path integral using the measure generated by Brownian motion. The natural generalization of Brownian motion is Lévy's motion. As long as the integral of path on Brownian trajectories lead to the standard Schrödinger equation, the path integral on Lévy trajectories leads to the fractional Schrödinger equation. This equation includes the fractional order derivative α instead of the second derivative $\alpha = 2$ in the standard Schrödinger equation. The results of fractional quantum mechanics have been discussed by several authors[4, 5, 6, 7, 8, 9, 10]. The objective of this work is solving the fractional Schrödinger equation for a linear potential based on Caputo-Fabrizio derivative [11].

2. FRACTIONAL SCHRÖDINGER EQUATION FOR A LINEAR POTENTIAL

The fractional Schrödinger equation with Caputo-Fabrizio derivative for a linear potential is given by

$$\left(\frac{\hbar^{2\gamma} D_{2\gamma}}{a_0^{2\gamma}}\right) {}^{CF}D_x^{2\gamma}\psi(x) + kx\psi(x) = -E\psi(x), \quad \frac{1}{2} < \gamma \leq 1, \tag{2.1}$$

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Key words and phrases. fractional Schrödinger equation; factional derivative; Caputo-Fabrizio derivative.

where $x = X/a_0$ is the reduced space coordinate, k is a constant coefficient and ${}^{CF}D_x^{2\gamma}$ is the Caputo-Fabrizio derivative defined as [11]

$$ {}^{CF}D_x^{\alpha} f(x) = \frac{1}{(1-\alpha)} \int_0^x \exp\left[\frac{-\alpha}{1-\alpha}(x-s)\right] f'(s)\mathrm{d}s, \quad 0 < \alpha \leq 1 \quad \& \quad x \leq 0. \tag{2.2} $$

We can rewrite the fractional Schrödinger equation (2.1) as follows

$$ \begin{aligned} {}^{CF}D_x^{2\gamma}\psi(x) &= \frac{-a_0^{2\gamma}}{\hbar^{2\gamma} D_{2\gamma}} (kx + E)\, \psi(x) \\ &= \frac{-a_0^{2\gamma} k}{2\hbar^{2\gamma} D_{2\gamma}} \left(x + \frac{E}{k}\right) \psi(x) \\ &= A_{2\gamma} (x + \varepsilon)\, \psi(x), \end{aligned} \tag{2.3} $$

where

$$ A_{2\gamma} = \frac{-a_0^{2\gamma}}{\hbar^{2\gamma} D_{2\gamma}} \quad \& \quad \varepsilon = \frac{E}{k}. \tag{2.4} $$

Assuming $\psi(x) = f(x)$ and $A_{2\gamma} (x + \varepsilon)\, \psi(x) = g(x)$, the formula (2.3) has the form

$$ {}^{CF}D_x^{2\gamma} f(x) = g(x). \tag{2.5} $$

First, let us take: ${}^{CF}D_x^{\gamma} f(x) = u(x)$ and ${}^{CF}D_x^{2\gamma} f(x) = {}^{CF}D_x^{\gamma} u(x) \equiv g(x)$, then, according to [12], we have

$$ u(x) = (1-\gamma)\,(g(x) - g(0)) + \gamma \int_0^x g(s)\mathrm{d}s + u(0), \tag{2.6} $$

and

$$ f(x) = (1-\gamma)\,(u(x) - u(0)) + \gamma \int_0^x u(s)\mathrm{d}s + f(0). \tag{2.7} $$

When we substitute (2.6) into (2.7), we obtain

$$ \begin{aligned} f(x) = (1-\gamma)^2 \,(g(x) - g(0)) + 2\gamma(1-\gamma) \int_0^x g(\tau)\mathrm{d}s + \gamma^2 \int_0^x \mathrm{d}s \int_0^s g(s)\mathrm{d}s \\ + \left[u(0) - (1-\gamma)g(0)\right] \gamma x + f(0). \end{aligned} \tag{2.8} $$

Taking the derivative twice, the last equation yields

$$ f''(x) = (1-\gamma)^2 g''(x) + 2\gamma(1-\gamma)g'(x) + \gamma^2 g(x). \tag{2.9} $$

By putting $f(x) = \psi(x)$ and $g(x) = A_{2\gamma} (x + \varepsilon)\, \psi(x)$, we find

$$ \begin{aligned} &\left[(1-\gamma)^2 x + (1-\gamma)^2 \varepsilon - \frac{1}{A_{2\gamma}}\right] \psi''(x) + \left[2\gamma(1-\gamma)x + 2\gamma(1-\gamma)\varepsilon + 2(1-\gamma)^2\right] \psi'(x) \\ &+ \left[\gamma^2 x + \gamma^2 \varepsilon + 2\gamma(1-\gamma)\right] \psi(x) = 0, \end{aligned} \tag{2.10} $$

or

$$ \left(\frac{c^2}{4}x + \frac{c^2}{4}\varepsilon + k\right)\psi''(x) + \left(cx + c\varepsilon + \frac{c^2}{2}\right)\psi'(x) + (x + c + \varepsilon)\psi(x) = 0, \tag{2.11} $$

or equivalently

$$ f_2(x)\psi''(x) + f_1(x)\psi'(x) + f_0(x)\psi(x) = 0, \tag{2.12} $$

where

$$k = -\frac{1}{\gamma^2 A_{2\gamma}}, \quad \text{and} \quad c = 2\left(\frac{1-\gamma}{\gamma}\right). \tag{2.13}$$

The solution of the equation (2.11) can be find as the following: put

$$\psi(x) = y(x)\exp(-\frac{1}{2}\int^x \frac{f_1(s)}{f_2(s)}ds). \tag{2.14}$$

The integral in the exponent is easy to perform and we get a novel equation governing $y(x)$

$$y''(x) + F(x)y(x) = 0, \tag{2.15}$$

where

$$F(x) = \frac{f_0(s)}{f_2(s)} - \frac{1}{4}\left(\frac{f_1(x)}{f_2(x)}\right)^2 - \frac{1}{2}\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{Ax+B}{(x+E)^2}, \tag{2.16}$$

such that

$$A = \frac{16k}{c^4}, \quad B = \frac{16k}{c^4}\left(\frac{1}{2}c + \varepsilon\right), \quad E = \varepsilon + \frac{4k}{c^2}. \tag{2.17}$$

The solution of the last differential equation is given by:

$$y(x) = \sqrt{x+E}\left[C_1 J_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E}) + C_2 Y_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E})\right]. \tag{2.18}$$

Then the final solution giving the wave function of the problem is given by

$$\psi(x) = \sqrt{x+E}\exp\left[-\frac{1}{2}\int^x \frac{f_1(s)}{f_2(s)}ds\right] \times \left[C_1 J_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E}) + C_2 Y_{\sqrt{1+4AE-4B}}(2\sqrt{A}\sqrt{x+E})\right], \tag{2.19}$$

where $J_\nu(X), Y_\nu(X)$ are the well known Bessel functions and C_1, C_2 are constants.

Conclusion

In conclusion, we have solved the fractional Schrödinger equation with Caputo-Fabrizio derivative for a linear potential. We have transformed this fractional differential equation to a second ordinary differential equation. The wave function then is easily calculated.

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Mean square error of the local linear estimation of the conditional mode function for functional data

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ABSTRACT

In this paper we treat the asymptotic mean square error and the rates of convergence of the estimator based on the local linear method of the conditional mode function. Under some general conditions, the expressions of the bias and variance are given. A practical application of data is introduced to demonstrate the usefulness of this estimation approach with respect to classical estimation of the conditional mode.

Keywords: Nonparametric local linear estimation,mean squared error, conditional mode function, functional variable.

2010 Mathematics Subject Classification. Primary 62G05, 62G20.

1 Introduction

In past years, substantial progress in computing capacity has allowed more and more complicated data to be collected and analysed. These wide data sets are accessible mainly through real-time monitoring, and computers can manage these databases efficiently. The issue of overelaborating statistics in the modeling of functional random variables has recognized growing holdings in late literature (see, for example, nonparametric context (**?**),(**?**).

Several multivariate statistical methods have been applied to functional data concerning parametric structures and a good overview of this subject can be given (**?**) or (**?**). New studies have recently been carried out to propose non-parametric methods which take functional data into account. The reader is referred to (**?**) and (**?**) for specialized monographs for a more detailed analysis on this subject.

In this functional component, (**?**) gets the prime results about the conditional mode estimation. They established the near maximum convergence of the kernel estimator in the i.i.d. scenario. (**?**) defined the dependent case. The monograph of (**?**) published an important set of statistical methods for nonparametric analysis of functional variables Recently the conditional mode using the k-Nearest Neighbors for independent functional data was done by ((**?**), the conditional mode using the k-NN method) for the dependency case was studied by (**?**).

Nonetheless, a local polynomial smoothing process is well known to have many advantages over the kernel approach (see, (**?**) and (**?**)). The former approach has improved properties, especially in terms of bias estimation. Most writers have taken into account the local linear smoothing in the practical data setting (**?**), (**?**), (**?**), (**?**) developed the first findings on regression function. Other studies on this topic were carried out, for example (**?**) developed a smoothing local linear regression operator estimation for independent results. In addition, (**?**) defined the near complete consistency of the conditional density local linear estimator when the explanatory variable is functional and the results are i.i.d. (**?**) have defined the asymptotic properties of the local linear estimator of the conditional cumulative distribution (almost complete convergence and convergence in mean square, with rates).

The rest of our paper is structured according to the following. We present our functional model in section 2, give simple notations and explain our assumptions. In section 3. We shall first state the key theoretical result of the paper on mean square convergence in subsection 3.1 and then,. Proofs are given in Section 4. in section 5 we prove the effectiveness of our study by a real data application and, We end the paper with Conclusion section 6.

2 Representation of the Model, Notation and Assumption

2.1 Model and estimator

Let's consider a sequence $(X_i; Y_i) i \geq 1$ of an independent and identically random pair according to the distribution of the pair (X, Y), both described in the same probability space (Ω, A, P) and taking their values in $\mathcal{F} \times \mathbb{R}$, where $(\mathcal{F}, d)$ is a semi-metric space.

In the following, x will be a fixed in $\mathcal{F}$; $\mathcal{N}_x$ resp. $\mathcal{N}_y$ will denote a fixed neighborhood of a fixed point x (resp. of y) and $\phi_x(r_1; r_2) = P(r_2 < \sigma(X; x) < r_1)$.

Moreover, f^x will denote the conditional density of the variable Y given X =x.

$$f^x(\widehat{\theta}) = \sup_{y \in \mathcal{S}} f^x(y). \tag{2.1}$$

We suppose that $f^x(.)$ has a only mode, noted by $\theta(x)$ assumed uniquely defined in the compact set S which is given by

$$f^x(\theta(x)) = \sup_{y \in \mathcal{S}} f^x(y). \tag{2.2}$$

$\theta(x)$ is a kernel estimator of the conditional mode which given as the random variable $\widehat{\theta}(x)$ that maximizes the kernel estimator $\widehat{f^x}(.)$ of $f^x(.)$.

$$\widehat{f^x}(\widehat{\theta}(x)) = \sup_{y \in \mathcal{S}} \widehat{f^x}(y). \tag{2.3}$$

The conditional distribution function $f^x(y)$ is calculated by quick functional local modeling (**?**) as the argmin value of an optimization problem for each $n \geq 1$,the following equation.

$$\widehat{f}^x(y) = \arg \min_{(a;b)\in\mathbb{R}^2} \sum_{i=1}^{n} (G(h_G^{-1}(y - Y_i)) - a - b\, \beta(X_i; x))^2 K(h_K^{-1}\delta(x; X_i)) \tag{2.4}$$

where $\beta(.;.)$ and $\delta(.;.)$ are locating functions defined from $\mathcal{F}^2$ into $\mathbb{R}$, such that:

$$\forall \xi \in \mathcal{F}; \beta(\xi; \xi) = 0 \;\; and \;\; d(.;.) = |\delta(.;.)|$$

and where the function K: kernel function, G: distribution function (df) and $h = h_K := h_{K,n}$ and $h_G = h_{G,n}$ are suites of positive real numbers, as n goes to infinity goes to zero . Clearly, the estimator $\widehat{a}$, given by (2.4), can be explicitly written as follows:

$$\widehat{f}^x(y) = \frac{\sum_{1\leq i,j\leq n} v_{ij}(x) G^{(1)}(h_G^{-1}(y - Y_i))}{h_G \sum_{1\leq i,j\leq n} v_{ij}(x)} \quad \forall y \in \mathrm{I\!R}. \tag{2.5}$$

then:
$G^{(1)}$ is the derivative of G, $v_{ij}(x) = \beta_i(\beta_i - \beta_j)K(h_K^{-1}\delta(x, X_i))$ with: $\beta_i = \beta(X_i, x)$ and the convention $0/0 = 0$;
We need more information, and simple hypotheses given below.

3 Notations and assumptions

Here and below, x (resp. y) will denote a fixed point in $(\mathcal{F}(resp.\mathbb{R})$, Then, we assume that our nonparametric model satisfies the following conditions:

(H1) For any $r > 0$, $\phi_x(r) := \phi_x(-r; r) > 0$. There exists a function $\chi_x(.)$ such that:

$$\forall t \in (-1, 1), \lim_{h_K\to 0} \frac{\phi_x(t\, h_K; h_K)}{\phi_x(h_K)} = \chi_x(t)$$

=

(H2) We denote, for any $l \in \{0, 2\}$ and $j = 0, 1$, the functions

$$\Psi_{l,j}(x, y) = \frac{\partial^l f^{x^j}(y)}{\partial y^l} \Psi_{l,j}(s) = \mathbb{E}[\Psi_{l,j}(X, y) - \Psi_{l,j}(x, y)|\beta(x, X) = s] \tag{3.1}$$

where $\Psi^1_{l,j}(0)$ and $\Psi^2_{l,j}(0)$ of the function $\Psi_{l,j}(.)$ exist and $g^{(k)}$ denotes the k^{th} order derivative of g.

(H3) The function $\beta(.,.)$ and $\delta(.,.)$ and are such that:

$$\forall z \in F, |\delta(x, z)| = d(x, z) \;\; and \;\; C_1|\delta(x, z)| \leq |\beta(x, z)| \leq C_2|\delta(x, z)|$$

where $C_1 > 0; C_2 > 0$.

$$\sup_{u\in B(x;r)} |\beta(u; x) - \delta(x; u)| = o(r)$$

and

$$h_K \int_{B(x;h_K)} \beta(u;x)dP(u) = o\left(\int_{B(x;h_K)} \beta^2(u;x)dP(u)\right)$$

where $B(x;r) = \{z \in \mathcal{F}/|\delta(z;x)| \leq r\}$ and $d\mathbb{P}$(x) is the cumulative distribution of X

(H4) The kernel K is a positive, differentiable function which is supported within $(-1;1)$ satisfies

$$K^2(1) - \int_{-1}^{1} (K^2(u))^{(1)} \chi_s(u) d(u) > 0$$

(H5) The kernel G is a differentiable function and $G^{(1)}$ is a positive, bounded, lipschitzian continuous function such that:

$$\int |t|^{b_2} G^{(1)}(t)dt < \infty \;\; and \;\; \int (G^{(1)})^2(t)dt < \infty \;\; and \;\; \int G^{(1)}(t)dt = 1$$

(H6) $\exists \alpha < \infty; \;\; f^x(y) \leq \alpha; \;\; \forall (x;y) \in \mathcal{F} \times \mathbb{R}$

(H7) The bandwidths h_K and h_G satisfy
$\lim_{n\to\infty} h_K = 0, \; \lim_{n\to\infty} h_G = 0$ width $\lim_{n\to\infty} n \; h_G^{(j)} \phi_x(h_K) = \infty$, for j=0,1.

(H8) The function $f^{(x_0)}$ is a 2-times continuously differentiable function with respect to y for all $x_0 \in \mathcal{N}_x$.

Some Remarks on the suppositions
Several Comments on the hypotheses: Hypothesis (H1) is the explanatory variable concentration property in small balls. The $\chi_x(.)$ function plays a crucial role in any asymptotic analysis, particularly for the term variance. The condition (H2) is used to monitor the regularity of our model's functional space, and this is important to determine the convergence rate bias concept. The assumption (H3) is a technical assumption. As established by (**?**). The assumption (H4) and (H5) on the Kernels K; G and $G^{(1)}$ are standard conditions for quadratic error determination for functional results. The hypotheses (H6) and (H7) are technical conditions and similar to those assumed in (**?**). (H8) The conditional density function is a slight regularity assumption.

4 Result

In this section we are going to state our theoretical results.

4.1 Mean Squared Convergence

We will need the following additional notation. C strictly positive generic constant. For all $(i,j) \in \{1,...,n\}^2$, we have

$K_i = K(h_K^{-1}\delta(X_i,x)), \; v_{ij} = v_{ij}(x),$

$G_j = H(h_G^{-1}(y - Y_j)), G_j^{(1)} = G_j^{(1)}(h_G^{-1}(y - Y_j)).$

Theorem 4.1. *Under assumptions (H1)-(H7)-(H8) we obtain*

$$\mathbb{E}[\widehat{\theta}(x)-\theta(x)] = B_{f;G}^2(x;y)h_G^4+B_{f;K}^2(x;y)h_K^4+\frac{V_{GK}^f(x;y)}{n\, h_G\phi_x(h_K)}+o(\sqrt{h^4G})+o(\sqrt{h^4K})+o\left(\sqrt{\frac{1}{n\, h_G\phi_x(h_K)}}\right) \quad (4.1)$$

Proof oh Theorem4.1:
the conditional density $f^x(.)$ is continuous see $(H8)$ we get

$$\forall\epsilon > 0,\ \exists\sigma(\epsilon) > 0, \forall y \in (\theta(x) - \xi, \theta(x) + \xi),\ |f^x(y) - f^x(\theta(x))| \leq \sigma(\epsilon) \Rightarrow |y - \theta(x)| \leq \epsilon.$$

By construction $\widehat{\theta}(x) \in (\theta(x) - \xi, \theta(x) + \xi)$ then

$$\forall\epsilon > 0,\ \exists\sigma(\epsilon) > 0,\ |f^x(\widehat{\theta}) - f^x(\theta(x))| \leq \sigma(\epsilon) \Rightarrow |y - \theta(x)| \leq \epsilon.$$

So that we arrive finally at

$$\exists\sigma(\epsilon) > 0,\ \mathbb{P}(|\widehat{\theta}(x) - \theta(x)| > \epsilon) \leq \mathbb{P}(|f^x(\widehat{\theta}(x)) - f^x(\theta(x))| > \delta(\epsilon)) \quad (4.2)$$

In the other case, it comes directly by the definition of $\theta(x)$ and $\widehat{\theta}(x)$ so we demonstrate this through simple empirical arguments

$$|f^x(\widehat{\theta}(x)) - f^x(\theta(x))| = 2\sup_{y\in\mathcal{S}} |\widehat{f^x}(y) - f^x(y)|. \quad (4.3)$$

Then we can write this by using a Taylor expansion of the f^x function:

$$f^x(\widehat{\theta}) = f^x(\theta) + \frac{1}{2!} f^{x2}(\theta^*)\,(\widehat{\theta} - \theta)^2\ ,$$

for some θ^* between θ and $\widehat{\theta}$ because (4.3), as along as we could be able to check that

$$\forall\tau > 0, \sum_{i=1}^{n} \mathbb{P}\left(f^{x(j)}(\theta^*) < \tau\right) < \infty. \quad (4.4)$$

we would have

$$(\widehat{\theta} - \theta)^2 = O\left(\sup_{y\in\mathcal{S}} |\widehat{f^x}(y) - f^x(y)|\right),\ almost\ \ completely, \quad (4.5)$$

This last is satisfied directly by using the continuity of $f^{x2}(.)$along with the result of the following lemma.

Lemma 4.2. *When the theorem 4.1 assumptions hold then we have*

$$\widehat{\theta} \to \theta \quad almost\ \ completely \quad (4.6)$$

And to prove the lemma, it is enough to combine 4.2 with 4.3 in order to achieve the stated result.

Theorem 4.3. *Under assumptions (H1)-(H7), we obtain*

$$\mathbb{E}[\widehat{f}^x(y) - f^x(y)] = B^2_{f;G}(x;y)h^4_G + B^2_{f;K}(x;y)h^4_K + \frac{V^f_{GK}(x;y)}{n\ h_G\phi_x(h_K)} + o(h^4_G) + o(h^4_K) + o(\frac{1}{n\ h_G\phi_x(h_K)}) \quad (4.7)$$

Theorem 4.4. *Under assumptions (H1)-(H7), we obtain*

$$\mathbb{E}[\widehat{f}^x(y) - f^x(y)] = B^2_{f;G}(x;y)h^4_G + B^2_{f;K}(x;y)h^4_K + \frac{V^f_{HK}(x;y)}{n\ h_G\phi_x(h_K)} + o(h^4 G) + o(h^4 K) + o(\frac{1}{n\ h_G\phi_x(h_K)}) \quad (4.8)$$

where $V^f_{GK}(x,y) = f^x(y)\left[\frac{\left(K^2(1) - \int_{-1}^1 (K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1) - \int_{-1}^1 (K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right]\int (G^{(1)}(t))^2\mathrm{d}t.$

with

$$\begin{array}{rcl} B_{f,G}(x,y) & = & \frac{1}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2 G^{(1)}(t)\mathrm{d}t \\ B_{f,K}(x,y) & = & \frac{1}{2}\Psi^{(2)}_{0,1}(0)\left[\frac{\left(K(1) - \int_{-1}^1 (u^2 K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1) - \int_{-1}^1 K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)}\right] \end{array}$$

We set

$$\widehat{f}^x_M(y) = \frac{1}{n(n-1)h_G\mathbb{E}[W_{12}(x)]}\sum_{1\leq i\neq j\leq n} W_{ij}(x)G^{(1)}(h_G^{-1}(y - Y_j))$$

and

$$\widehat{f}_S(x) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(x)]}\sum_{1\leq i\neq j\leq n} W_{ij}(x)$$

then

$$\widehat{f}^x(y) = \frac{\widehat{f}^x_M(y)}{\widehat{f}_S(x)}.$$

The proof of Theorem 4.4 can be deduced from the following intermediates results.

Lemma 4.5. *Under the hypotheses of Theorem 4.4, we get*

$$\mathbb{E}\left[\widehat{f}^x_N(y)\right] - f^x(y) = B_{f,G}(x,y)h^2_G + B_{f,K}(x,y)h^2_K + o(h^2_G) + o(h^2_K).$$

Lemma 4.6. *Under the hypotheses of Theorem 4.4, we have*

$$Var\left[\widehat{f}^x_N(y)\right] = \frac{V^f_{GK}(x,y)}{nh_H\phi_x(h_K)} + o\left(\frac{1}{nh_G\phi_x(h_k)}\right).$$

Lemma 4.7. *Under the hypotheses of Theorem 4.4, we get*

$$Cov(\widehat{f}^x_M(y), \widehat{f}_S(x)) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Lemma 4.8. *Under the hypotheses of Theorem 4.4, we have*

$$Var\left[\widehat{f}_D(x)\right] = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

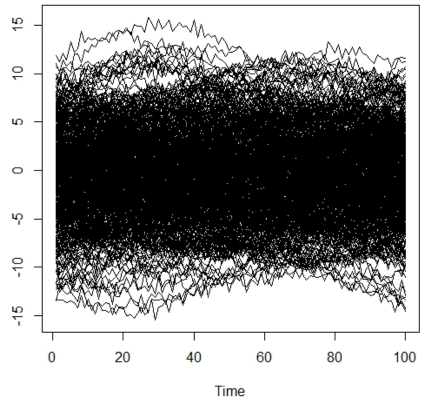

Figure 1: Curves X_i

5 Simulation study: Real data application

Conditional density is a very useful tool for describing the relationship between two random variables. In this chapter we will illustrate the estimation of this non-parametric model by the local linear method. The main objective is to show using simulated real data the applicability of this method in the functional case. We will illustrate the conditional mode as forecasting tools closely linked to the estimation of the conditional density. Our aim is to show the superiority of the method of local linear on the kernel method.

For this illustration, we consider functional observations generated using the following processes;

$$X_i(t) = cos((1 - v_i)t) + sin(v_i t) + b_i, \quad \forall t \in [0, \pi] \;\; and \;\; i = 1, 2, \cdots, n$$

or v_i (resp. b_i) is of law $U(0, 1)$ (resp. is of $N(1, 0.4)$). We suppose that these curves are observed on a discretization grid of 100 points in the interval $[0, \pi]$. These functional variables are represented on the following graph

For response variables Y_i, we consider the regression model

$$Y = L(X) + \epsilon$$

and ϵ is of normal law $N(0, 0.3)$.
where

$$L(x) = 4 \exp\left(\frac{1}{1 + \int_0^{\pi} |x(t)|^2 dt}\right).$$

Our purpose in this illustration is to show the utility of conditional density in a forecast context. To this effect, we divide our observations on two learning sample packages $(X_i, Y_i)_{i=1,\cdots 500}$

and test sample and $(X_i, Y_i)_{i=501,\cdots 550}$. For the latter, we assume that the response values are unknowns and we're going to approximate it by $\theta(X_i)$ defined in 2.2 and estimated by $\hat{\theta}(X_i)$ defined in 2.3.
To verify the effectiveness of this model, in this forecast analysis we compare the quantities $(Y_i)_{i=501,\cdots 550}$ with $\hat{\theta}(X_i)_{i=501,\cdots 550}$.

$$Error(h_k, h_G) = |Y_i - \hat{\theta}(X_i)|. \tag{5.1}$$

In conclusion, we can say that the estimation of the conditional mode in use reading the local linear method is very effective as a model of forecast and we have found that this model is easy to handle and that our programs give results quickly, so the MSE of local linear conditional mode estimation is 0.30 and the MSE of the classical mode estimation used in Ferraty et al. (**?**) is 2.97524, this result exhibit the effectiveness of our study.

6 Proofs

The proofs are structured in the following way Theorem 4.4 introduces the conditional density estimator's mean square error. To demonstrate this theorem, we require 4.5-4.8 lemmas to test it.

Proof of Theorem 4.4. We begin by computing the bias and the variance of $\widehat{f^x}(y)$. We have

$$\mathbb{E}\left[\widehat{f^x}(y) - f^x(y)\right]^2 = \left[\mathbb{E}\left[\widehat{f^x}(y)\right] - f^x(y)\right]^2 + Var\left[\widehat{f^x}(y)\right]. \tag{6.1}$$

By simple calculations, we get

$$\begin{aligned}\widehat{f^x}(y) - f^x(y) &= \left(\widehat{f^x_M}(y) - f^x(y)\right) - \left(\widehat{f^x_M}(y) - \mathbb{E}[\widehat{f^x_M}(y)]\right)\left(\widehat{f_S}(x) - 1\right) \\ &\quad -\mathbb{E}[\widehat{f^x_M}(y)]\left(\widehat{f_S}(x) - 1\right) + \left(\widehat{f_S}(x) - 1\right)^2 \widehat{f^x}(y).\end{aligned}$$

From that fact that $\mathbb{E}[\widehat{f_S}(x)] = 1$, we deduce that:

$$\begin{aligned}\mathbb{E}\left[\widehat{f^x}(y)\right] - f^x(y) &= \left(\mathbb{E}[\widehat{f^x_M}(y)] - f^x(y)\right) - Cov\left(\widehat{f^x_M}(y), \widehat{f_S}(x)\right) \\ &\quad +\mathbb{E}\left[\left(\widehat{f_S}(x) - \mathbb{E}[\widehat{f_S}(x)]\right)^2 \widehat{f^x}(y)\right].\end{aligned}$$

Because the $G^{(1)}$ kernel is bounded, we may bind $\widehat{f^x}(y)$ by a $Co > 0$constant, so $\widehat{f^x}(y) \leq Co/hG$. Therefore

$$\begin{aligned}\mathbb{E}\left[\widehat{f^x}(y)\right] - f^x(y) &= \left(\mathbb{E}[\widehat{f^x_M}(y)] - f^x(y)\right) - Cov\left(\widehat{f^x_M}(y), \widehat{f_S}(x)\right) \\ &\quad +Var\left[\widehat{f_S}(x)\right] O(h_G^{-1}).\end{aligned}$$

Now the variance word in (6.1) is by (**?**)

$$\begin{aligned}Var\left[\widehat{f^x}(y)\right] &= Var\left[\widehat{f^x_M}(y)\right] - 2\mathbb{E}[\widehat{f^x_M}(y)]Cov\left(\widehat{f^x_N}(y), \widehat{f_D}(x)\right) \\ &\quad + \left(\mathbb{E}[\widehat{f^x_M}(y)]\right)^2 Var\left(\widehat{f_S}(x)\right) o\left(\frac{1}{nh_G\phi_x(h_K)}\right).\blacksquare\end{aligned}$$

Proof of Lemma 4.5. We've got

$$\mathbb{E}[\widehat{f}_M^x(y)] = \frac{1}{h_G \mathbb{E}[v_{12}]} \mathbb{E}\left[v_{12}\mathbb{E}[G_2^{(1)}|X_2]\right]. \tag{6.2}$$

Using an expansion of Taylor and considering $(H5)$, we get

$$\mathbb{E}[G_2^{(1)}|X_2] = f^{X_2}(y) + \frac{h_G^2}{2}\left(\int t^2 G^{(1)}(t)\mathrm{d}t\right)\frac{\partial^2 f^{X_2}(y)}{\partial y^2} + o(h_G^2).$$

The following is rewritable as

$$\mathbb{E}[G_2^{(1)}|X_2] = \psi_{0,1}(X_2, y) + \frac{h_G^2}{2}\left(\int t^2 G^{(1)}(t)dt\right)\psi_{2,1}(X_2, y) + o(h_G^2).$$

Hence we attain from (6.2)

$$\mathbb{E}\left[\widehat{f}_M^x(y)\right] = \frac{1}{E[v_{12}]}\left(\mathbb{E}\left[v_{12}\psi_{0,1}(X_2,y)\right] + \frac{h_G^2}{2}\left(\int t^2 G^{(1)}(t)\mathrm{d}t\right)\mathbb{E}\left[v_{12}\psi_{2,1}(X_2,y)\right] + o(h_G^2)\right).$$

We should shot that for $l \in \{0, 2\}$, as shown by (**?**)

$$\begin{aligned}\mathbb{E}[v_{12}\psi_{l,1}(X_2,y)] &= \mathbb{E}[v_{12}(\psi_{l,1}(X_2,y) - \psi_{l,1}(x,y))] + \psi_{l,1}(x,y)\mathbb{E}[v_{12}] \\ &= \mathbb{E}[v_{12}\Psi_{l,1}(\beta(X_2,x))] + \psi_{l,1}(x,y)\mathbb{E}[v_{12}].\end{aligned}$$

By Seeing $\Psi_{l,1}(0) = 0$ and $\mathbb{E}\left[\beta(X_2,x)v_{12}\right] = 0$, we get

$$\mathbb{E}\left[v_{12}\psi_{l,1}(X_2,y)\right] = \psi_{l,1}(x,y)\mathbb{E}[v_{12}] + \frac{1}{2}\Psi_{l,1}^{(2)}(0)\mathbb{E}\left[\beta^2(X_2,x)v_{12}\right] + o(\mathbb{E}\left[\beta^2(X_2,x)v_{12}\right]).$$

That,

$$\begin{aligned}\mathbb{E}\left[\widehat{f}_M^x(y)\right] &= f^x(y) + \frac{h_G^2}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2 G^{(1)}(t)\mathrm{d}t + o\left(h_G^2 \frac{\mathbb{E}\left[\beta^2(X_2,x)v_{12}\right]}{\mathbb{E}[W_{12}]}\right) \\ &\quad + \Psi_{0,1}^{(1)}(0)\frac{\mathbb{E}\left[\beta^2(X_2,x)v_{12}\right]}{2\mathbb{E}[v_{12}]} + o\left(\frac{\mathbb{E}\left[\beta^2(X_2,x)v_{12}\right]}{\mathbb{E}[v_{12}]}\right).\end{aligned}$$

The two quantities $\mathbb{E}\left[\beta(x,X_2)^2 v_{12}\right]$ and $\mathbb{E}[v_{12}]$ are based on the asymptotic evaluation of $\mathbb{E}[K_1^a\beta_1^b]$ (see (**?**) for more details). To do that, first we treat the case $b = 1$ and $a > 0$. For this case, we use the last part of $(H3)$ and $(H4)$, to get

$$h_K\mathbb{E}[K_1^a\beta_1] = o\left(\int_{B(x,h_K)} \beta^2(u,x)\mathrm{d}P_X(u)\right) = o(h_K^2\phi_x(h_K)).$$

we obtain

$$\mathbb{E}[K_1^a\beta_1] = o(h_K\phi_x(h_K)). \tag{6.3}$$

At the other side, we have $b > 1$for all, and after simplifying the expressions,

$$\mathbb{E}[K_1^a\beta_1^b] = \mathbb{E}[K_1^a\delta^b(x,X)] + o(h_K^b\phi_x(h_K)).$$

Concerning the first term, we write

$$\begin{aligned}h_K^{-b}\mathbb{E}[K_1^a\delta^b] &= \int v^b K^a(v)\mathrm{d}P_X^{h_K^{-1}\delta(x,X)}(p) \\ &= \phi_x(h_K)\left(K(1) - \int_{-1}^{1}(u^bK^a(u))^{(1)}\frac{\phi_x(uh_K,h_K)}{\phi_x(h_K)}\mathrm{d}u\right).\end{aligned}$$

Then, under assumptions $(H1)$, we get

$$\mathbb{E}[K_1^a\beta_1^b] = h_K^b\phi_x(h_K)\left(K(1) - \int_{-1}^{1}(u^bK^a(u))^{(1)}\chi_x(u)\mathrm{d}u\right) + o(h_K^b\phi_x(h_K)). \qquad (6.4)$$

So,

$$\frac{\mathbb{E}[\beta^2(X_2,x)v_{12}]}{\mathbb{E}[v_{12}]} = h_K^2\left(\frac{K(1) - \int_{-1}^{1}(u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u}{K(1) - \int_{-1}^{1}(K^{(1)}(u)\chi_x(u)\mathrm{d}u}\right) + o(h_K^2).$$

So,

$$\begin{aligned}\mathbb{E}\left[\widehat{f}_M^x(y)\right] &= f^x(y) + \frac{h_G^2}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2H^{(1)}(t)\mathrm{d}t + o(h_G^2)\\ &\quad + \frac{h_K^2}{2}\Psi_{0,1}^{(2)}(0)\frac{\left(K(1) - \int_{-1}^{1}(u^2K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1) - \int_{-1}^{1}K^{(1)}(u)\chi_x(u)\mathrm{d}u\right)} + o(h_K^2).\blacksquare\end{aligned}$$

Proof of Lemma 4.6.

$$Var\left(\widehat{f}_M^x(y)\right) = \frac{1}{(n(n-1)h_G(\mathbb{E}[v_{12}]))^2}Var\left(\sum_{1\le i\ne j\le n} v_{ij}G_j^{(1)}\right)$$

$$\begin{aligned}= \frac{1}{(n(n-1)h_H(\mathbb{E}[W_{12}]))^2}\Big[&n(n-1)\mathbb{E}[W_{12}^2(H_2^{(1)})^2] + n(n-1)\mathbb{E}[W_{12}W_{21}H_2^{(1)}H_1^{(1)}]\\ &+n(n-1)(n-2)\mathbb{E}[v_{12}v_{13}G_2^{(1)}G_3^{(1)}] + n(n-1)(n-2)\mathbb{E}[v_{12}v_{23}G_2^{(1)}G_3^{(1)}]\\ &-n(n-1)(4n-6)\mathbb{E}[v_{12}G_2^{(1)}]^2\Big].\end{aligned} \qquad (6.5)$$

After some simple calculation we get

$$\begin{cases}\mathbb{E}[v_{12}^2G_2^{(1)}] = O(h_K^4h_G\phi_x^2(h_K)), \quad \mathbb{E}[V_{12}W_{21}G_2^{(1)}H_1^{(1)}] = O(h_K^4h_G^2\phi_x^2(h_K)),\\ \mathbb{E}[v_{12}v_{13}G_2^{(1)}G_3^{(1)}] = \mathbb{E}[v_{12}v_{31}G_2^{(1)}G_1^{(1)}] = \mathbb{E}[v_{12}v_{23}G_2^{(1)}G_3^{(1)}] = O(h_K^4h_G^2\phi_x^3(h_K)),\\ \mathbb{E}[v_{12}v_{32}(G_2^{(1)})^2] = \mathbb{E}^2[\beta_1^2K_1]\mathbb{E}[K_1^2(G_1^{(1)})^2] + o(h_K^4h_G\phi_x^3(h_K)).\end{cases}$$

Obviously, in the final cases, the above term is the leading term, which can be tested in (6.5)

$$\frac{(n-2)}{n(n-1)(h_G\mathbb{E}[v_{12}])^2}\mathbb{E}^2[\beta_1^2K_1]\mathbb{E}[K_1^2(G_1^{(1)})^2]$$

So it suffices to write after the same steps as in the previous Lemma

$$Var\left(\widehat{f}_M^x(y)\right) = \frac{\mathbb{E}[K_1^2(G_1^{(1)})^2]}{n(h_G\mathbb{E}[K_1])^2} + o\left(\frac{1}{nh_G\phi_x(h_K)}\right). \qquad (6.6)$$

Thus, by the change of variables $t = h_G^{-1}(r-z)$, we get

$$\mathbb{E}[K_1^2(G_1^{(1)})^2] = \mathbb{E}[K_1^2\mathbb{E}((G_1^{(1)})^2|X_1)]$$

and

$$\mathbb{E}((G_1^{(1)})^2|X_1) = h_G\int(G^{(1)})^2(t)f^{X_1}(r-h_Gt)\mathrm{d}t.$$

Then by order 1 of $f^{X_1}(\cdot)$, with Taylor's expansion we get

$$f^{X_1}(r-h_Gt) = f^{X_1}(y) + O(h_G) = f^{X_1}(r) + O(1).$$

Now, from (6.6) it follows that:

$$\mathbb{E}[K_1^2(G_1^{(1)})^2] = h_G \int (G^{(1)})^2(t)\mathrm{d}t\mathbb{E}\big[K_1^2 f^X(y)\big] + o(h_G\mathbb{E}[K_1^2]).$$

Once more, we get Lemma 4.5 by the same measures in proof

$$\mathbb{E}\big[K_1^2 f^{X_1}(y)\big] = f^x(y)\mathbb{E}[K_1^2] + o(\mathbb{E}[K_1^2])$$

That would imply:

$$\mathbb{E}[K_1^2(G_1^{(1)})^2] = h_G f^x(y)\mathbb{E}[K_1^2]\int (G^{(1)})^2(t)\mathrm{d}t + o(h_G\mathbb{E}[K_1^2]). \tag{6.7}$$

Therefore we get from (6.4), (6.6) and (6.7) the

$$Var\left(\widehat{f}_M^x(y)\right) = \frac{f^x(y)}{nh_G\phi_x(h_K)}\left(\int G^{(1)}(t)^2\mathrm{d}t\right)\left[\frac{\left(K^2(1)-\int_{-1}^1(K^2(u))^{(1)}\chi_x(u)\mathrm{d}u\right)}{\left(K(1)-\int_{-1}^1(K(u))^{(1)}\chi_x(u)\mathrm{d}u\right)^2}\right]$$
$$+o\left(\frac{1}{nh_G\phi_x(h_K)}\right).\blacksquare$$

Proof of Lemma 4.7. We had it through easy calculations

$$\begin{aligned}
Cov\left(\widehat{f}_M^x(y),\widehat{f}_S(x)\right) &= \frac{1}{(n(n-1)h_G(\mathbb{E}[v_{12}]))^2}Cov\Big(\sum_{1\le i\neq j\le n} v_{ij}G_j^{(1)}, \sum_{1\le i'\neq j'\le n} v_{i'j'}\Big)\\
&= \frac{1}{(n(n-1)h_G(\mathbb{E}[v_{12}]))^2}\Big[n(n-1)\mathbb{E}[v_{12}^2G_1^{(1)}] + n(n-1)\mathbb{E}[v_{12}v_{21}v_2^{(1)}]\\
&\quad +n(n-1)(n-2)\mathbb{E}[v_{12}v_{13}G_2^{(1)}] + n(n-1)(n-2)\mathbb{E}[v_{12}v_{23}G_2^{(1)}]\\
&\quad +n(n-1)(n-2)\mathbb{E}[v_{12}v_{31}G_2^{(1)}] + n(n-1)(n-2)\mathbb{E}[v_{12}v_{32}G_2^{(1)}]\\
&\quad -n(n-1)(4n-6)(\mathbb{E}[v_{12}G_2^{(1)}]\mathbb{E}[v_{12}]\Big].
\end{aligned}$$

By direct manipulations, we get

$$\begin{cases}
\mathbb{E}[v_{12}^2G_2^{(1)}] = \mathbb{E}[v_{12}v_{21}G_2^{(1)}] = O(h_K^4h_G\phi_x^2(h_K)),\\
\mathbb{E}[v_{12}v_{13}G_2^{(1)}] = \mathbb{E}[v_{12}v_{31}G_2^{(1)}] = O(h_K^4h_G\phi_x^3(h_K)),\\
\mathbb{E}[v_{12}W_{23}H_2^{(1)}] = \mathbb{E}[W_{12}W_{32}H_2^{(1)}] = O(h_K^4h_G\phi_x^3(h_K)).
\end{cases}$$

Since $\mathbb{E}[v_{12}] = O(h_K^2\phi_x^2(h_K))$, we obtain

$$Cov\left(\widehat{f}_M^x(y),\widehat{f}_S(x)\right) = O\left(\frac{1}{n\phi_x(h_K)}\right).\blacksquare$$

Proof of Lemma 4.8. The presentation of this finding follows the lines of the preceding lemma proof, step by step, by replacing $H^{(1)}$ with 1. Accordingly,

$$\begin{aligned}
Var(\widehat{f}_S^x) &= \frac{1}{(n(n-1)\mathbb{E}[v_{12}])^2}Var\left(\sum_{1\le i\neq j\le n} W_{ij}\right)\\
&= \frac{1}{(n(n-1)\mathbb{E}[v_{12}]))^2}\left(n(n-1)\mathbb{E}[v_{12}^2] + n(n-1)\mathbb{E}[v_{12}v_{21}]\right.\\
&\quad +n(n-1)(n-2)\mathbb{E}[v_{12}v_{13}] + n(n-1)(n-2)\mathbb{E}[v_{12}v_{23}]\\
&\quad +n(n-1)(n-2)\mathbb{E}[v_{12}v_{31}] + n(n-1)(n-2)\mathbb{E}[v_{12}v_{32}]\\
&\quad \left.-n(n-1)(4n-6)(\mathbb{E}[v_{12}])^2\right).
\end{aligned}$$

We always get through simple manipulations

$$\begin{cases} \mathbb{E}[v_{12}^2] = \mathbb{E}[v_{12}v_{21}] = O(h_K^4\phi_x^2(h_K)), \\ \mathbb{E}[v_{12}v_{13}] = \mathbb{E}[v_{12}v_{31}] = O(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[v_{12}v_{23}] = \mathbb{E}[v_{12}v_{32}] = O(h_K^4\phi_x^3(h_K)). \end{cases}$$

So, we obtain

$$Var\left(\widehat{f}_S^x\right) = O\left(\frac{1}{n\phi_x(h_K)}\right). \blacksquare$$

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EXISTENCE OF SOLUTIONS FOR BOUNDARY VALUE PROBLEMS OF DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS VIA FIXED POINT THEORY

MAROUA MENECEUR AND SAID BELOUL

ABSTRACT. In this study, we discuss the existence of the solution for the following a boundary value problem of differential inclusions:

$$\begin{cases} x''(t) \in F(t, x(t)), \ 0 \leq t \leq 1 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 g(s, x(s))ds \end{cases} \tag{1}$$

where $t \in J = [0, 1]$, $0 < a < b$, $F : J \times \mathbb{R} \times \mathbb{R} \longrightarrow K(\mathbb{R})$ is a lower semi continuous multivalued function and $g : J \times \mathbb{R} \to \mathbb{R}$ a continuous function.
The used technique is based on multivalued fixed point for θ-contractions in ordered metric spaces.

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Global solution for the Fisher equation

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Abstract. This paper is devoted to existence for global solutions of the Fisher equation in bounded domains. We prove the existence a global solution for the equation with a homegeneous Dirichlet condition by Faedo-Galerkin's method.
Keywords: global solution, local solution, Faedo-Glerkin method.

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1 Introduction

This model describes the advance of a population in space with the following equation, which we denote the Fisher-Kolmogorov-Petrovsky-Piskunov (F-KPP) equation[6]

$$u_t - Du_{xx} = \mu u(K-u) \;\; in \;]0,T[\times]0,K[$$

where x denotes position, t denotes time, and $u(x,t)$ is the population density. In this model, population expansion arises from a balance between diffusion of individuals in space (with diffusion constant D) and local growth (with maximum population density or carrying capacity K and linear growth rate αK). With the homogeneous boundary Dirichlet conditions

$$u(t,0) = u(t,K) = 0 \;\; for \; t \in]0,T[$$

and

$$u(0,x) = u_0(x) \;\; for \; x \in]0,K[.$$

This equation has extremely broad biological relevance Murray 2004[13]; Ackland et al. 2007[1]; Barrett-Freeman et al. 2008[3]; Rouzine et al. 2008[15]; Greulich et al. 2012[8] and is also important in other fields, including applied mathematics van Saarloos 2003[18],statistical physics and computer science (Majumdar and Krapivsky 2002, 2003)[12].

This equation has important examples include parasites carried by an invading population, which may have catastrophic consequences for the native species (Prenter et al. 2004[14]; Bar-David et al. 2006[2]) or in some cases be used as a means to control the invaders (Fagan et al. 2002[5]), and horizontal gene transfer within spatially structured bacterial communities, which presents dangers for the spread of antibiotic resistance, but also opportunities for bioremediation Fox et al. 2008[7].

In this paper, we consider the model analysis of the Fisher equation

$$u_t - \Delta u = \mu u(1-u) \;\; in \;]0,T[\times\Omega \tag{1.1}$$

with homogeneous boundary Dirichlet conditions. Here,Ω is a bounded open set in $\mathbb{R}^n$, $n = 1,2,$ or 3, $[0,T]$ is a finite time interval, . Note that in the Fisher equation enters not merely as an inhomogeneous source term, but in the more complicated way indicated in (1.1). The plan of this paper is as follows. In Section 2 we introduce function spaces, define a weak formulation and recall some results from ODP and ODE theories. In Section 3 we establish the existence of local solution.In section 4 wr prove a maximum principale under suitable on u_0. In Section 5 we prove the existence and uniqueness of solution.

2 Problem statement and preliminaries

In this paper,we consider the problem we wish to study is a multidimensional version of the Fisher equation (1.1) with a Dirichlet boundary condition

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \mu u(1-u) & in\,]0,T[\times\Omega \\ u = 0 & on\ \Gamma =]0,T[\times\partial\Omega \\ u(0,x) = u_0(x) & in\ \Omega \end{cases} \tag{2.1}$$

In this section, we will introduce function spaces and introduce a weak formulation of (2.1). We will also quote some relevant ODE results.

2.1 Function spaces and problem statement

In the sequel, Ω is a bounded, open bounded domain in $\mathbb{R}^d, d = 1,2,3$, with a smooth boundary $\partial\Omega$. $H^s(\Omega)$ for $s \in \mathbb{R}$ denotes the standard real Sobolev space of order s with its norm denoted by $\|.\|_{H^s(\Omega)}$ .We use the convention $H^0(\Omega) = L^2(\Omega)$. For a $p \in [1,\infty[$, an open interval $]a,b[\subset \mathbb{R}$ and a Banach space B with the norm $\|.\|_B$, we denote by $L^p(a,b;B)$ the set of measurable functions $u :]a,b[\to B$ such that $t \to \|u(t)\|_{B)}$ belongs to $L^p(a,b)$. The norm on $L^p(a,b;B)$ is defined by

$$\|u(t)\|_B = \begin{cases} \left(\int_a^b \|u(t)\|_B^p \right)^{1/p} & if\ \ p < \infty \\ Ess \sup_{t\in]a,b[} \|u(t)\|_B & if\ \ p = \infty \end{cases}$$

denote by $C([a,b];B)$ the set of continuous functions $u : [a,b] \to B$ with the norm $\|u(t)\|_{C([a,b];B)} = max_{t\in[a,b]}\|u(t)\|_B$.We introduce

$$W(a,b) = \left\{ u \setminus u \in L^2(a,b;H_0^1(\Omega), \frac{\partial u}{\partial t} = u' \in L^2(a,b;H_0^{-1}(\Omega) \right\}$$

where u' is taken in the sense of distribution. The norm on $W(a,b)$ is defined by

$$\|u\|_{W(a,b)} = (\|u\|_{L^2(a,b;H_0^1(\Omega))} + (\|u\|_{L^2(a,b;H_0^{-1})}(\Omega))^{1/2}, \ \ \forall \in W(a,b)$$

The duality paring between a Banach space B and its dual will be denoted by $\langle .,. \rangle$. The $L^2(\Omega)$ inner product is denoted by $(.,.)$, i.e., for $p,q \in L^2(\Omega), (p,q) = \int_\Omega p.qd\Omega$. Also,$C, \widetilde{C}, C_1, C_2$, etc. denote positive constants whose values change with context. A solution for (3.2) is defined as a solution of the following weak formulation: given , find a $u \in W(0,T)$ such that

$$\langle u'(t), \phi \rangle + \langle \nabla u(t), \nabla \phi \rangle = \langle \mu u(t)(1-u(t)), \phi \rangle \ \ \forall \phi \in H_0^1(\Omega),\ a.e.\, t \in]0,T[\tag{2.2}$$

and

$$u(0,x) = u_0(x) \tag{2.3}$$

Lemma 2.1. *Let V, H, V' be three Hilbert spaces such that $V \subset H = H' \subset V'$, where V' is the dual of V . If a function u satisfies that and $u \in L^2(t_0, t_1; V)$ and $u' \in L^2(t_0, t_1; V')$, then u is almost everywhere equal to a function continuous from $[t_0, t_1]$ into H and we have the following equality, which holds in the scalar distributional sense on (t_0, t_1):*

$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u', u\rangle$

Proof. See [16]. □

Lemma 2.2. *Let $(u_k) \subset H$ such that $u_k \rightharpoonup u$. Then*

- *(u_k) is bounded.*
- *$\|u_k\| \leq lim\,inf_{k\to\infty}\|u_k\|$.*

Proof. See [16]. □

Theorem 2.3. *[Poincaré inequalitie] Let Ω is ouver and bounded that then exist a constant C , depending on Ω and p, such that.*

$$\|u\|_{L^p} \leq \|\nabla u\|_{L^p} \quad \forall u \in W_0^{1.p}(\Omega) \;\; (1 \leq p < \infty)$$

Lemma 2.4 (Hölder's inequality). *We fix $p, q \in [1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$ Suppose Ω ouvert $\in \mathbb{R}^n$and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$: Then $f.g \in L^1(\Omega)$ and*

$$\int_\Omega |fg|dx \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}$$

Lemma 2.5 (young's inequality). *We fix $p, q, r \in [1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ Suppose Ω ouvert $\in \mathbb{R}^n$and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$: Then*

$$\|f * g\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}$$

2.2 Relevant ODE results

In this subsection, we recall some relevants results concerning the existence of and estimates for solutions of ODE systems. The first lemma is about the existence of a maximal solution on a rectangular region.

Lemma 2.6. *Let real numbers t_0, y_0, $a > 0$, $b > 0$ be given and set*
$R_0 = [t_0, t_0 + a][y_0.b, y_0 + b]$.
Assume that $g \in C(R_0; \mathbb{R})$. Denote

$$M = \max_{(t,y)\in R_0} |g(t, y)| \;\; and \;\; \tau = \min\left\{a, \frac{b}{2M + b}\right\}$$

Then the initial value problem

$$\frac{du}{dt} = g(t, y), \;\; g(t_0) = u_0 \tag{2.4}$$

has a maximal solution $\overline{y}(t)$ on $[t_0, t_0 + \tau]$, i.e., every solution $y = y(t)$ of

$$y' = g(t, y) \;\; on \; [t_0, t_0 + \tau], \;\; y(t_0) = u_0$$

satisfies $y(t) \leq \overline{u}(t)$ on $[t_0, t_0 + \tau]$

Proof. See [11] . □

The second lemma is about the existence of a maximal solution on a strip.

Lemma 2.7. *Let t_0, y_0, $a > 0$ be given. Assume that $g \in C([t_0, t_0 + a] \times \mathbb{R}; \mathbb{R})$. Then there exists a $\tau > 0$ such that (2.4) has a maximal solution u(t) on $[t_0, t_0 + \tau]$, i.e., every solution $y = y(t)$ of*

$$y' = g(t, y) \ \text{ on } [t_0, t_0 + \tau], \quad y(t_0) = y_0$$

satisfies $y(t) \leq \overline{y}(t)$ on $[t_0, t_0 + \tau]$

Proof. This is a direct consequence of the previous lemma (e.g., by fixing a finite rectangle on the strip). □

The third lemma is about the extension of solutions of a system of ODEs over a maximal interval of existence.

Lemma 2.8. *Assume that $G \in C([t_0, t_0 + a] \times \mathbb{R}^d; \mathbb{R}^d)$ where $d \geq 1$ is an integer. Let $y = y(t)$ be a solution, on a right maximal interval J, of $y' = G(t, y)$ with a given $y(t_0)$. Then either $J = [t_0, t_0 + a]$ or $J = [t_0, t_0 + \delta)$, $\delta < a$, and $\|y(t)\|_{\mathbb{R}^d} \to \infty$ as $t \to t_0 + \delta$.*

Proof. See [7]. □

The next lemma is about an estimate, in terms of the maximal solution, for solutions of an integral inequality.

Lemma 2.9. *Assume that $g \in C([t_0, t_0 + a] \times \mathbb{R}; \mathbb{R})$, $g(t, y)$ is nondecreasing in u for each $t \in [t_0, t_0 + a]$ and a maximal solution $y(t)$ of (2.4) exists on $[t_0, t_0 + a]$. Assume further that $h \in C([t_0, t_0 + a]; \mathbb{R})$,$h(t_0) \leq u_0$, and*

$$h(t) \leq h(t_0) + \int_{t_0}^{t} g(s, h(s))ds \quad \forall t \in [t_0, t_0 + a[.$$

Then

$$h(t) \leq \overline{u}(t) \quad \forall t \in [t_0, t_0 + a[.$$

Proof. See [11]. □

3 Local existence

In this section, we study the existence of a local solution for the weak formulation (2.2) and (2.3).We first recall the following compact embedding result:

Lemma 3.1. *Let X_0, X, X_1 be three Banach Spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$ and $X_0 \hookrightarrow\hookrightarrow X$, let $1 < p, q < \infty$ and*

$$W = \{v(t) \in L^p(t_0, t_1; X_0) : \frac{du(t)}{dt} L^q(t_0, t_1; X_1)\}.$$

Then $W \hookrightarrow\hookrightarrow L^p(t_0, t_1; X)$.

Proof. See [11]. □

We are now in a position to state and prove the local existence results.

Theorem 3.2. *Assume that $b_0 \in L^2(\Omega)$ and $t_0 \in [0,T[$. Then there exists a $t_1 = t_1(t_0, b_0) \in]t_0, T[$ and a $u \in W(t_0,t_1)$ satisfying*

$$\langle u'(t), \phi\rangle + \langle \nabla u(t), \nabla\phi\rangle = \langle \mu u(t)(1-u(t)), \phi\rangle \ \ \forall\phi \in H_0^1(\Omega), \ a.e.\, t \in]0,T[\tag{3.1}$$

and

$$u(t_0) = b_0(x) \tag{3.2}$$

Proof. we shall use the so-called Faedo-Galerkin method. We divide the proof into three steps. In Step 1 solution of the approximate problem. In Step 2 a priori estimates for u_m are derived. In Step 3 passage to limits, derive the regularity of u' and justify the initial condition.

Step 1: Solution of the approximate problem:
Since $H_0^1(\Omega)$ is separable, there exists a basis $\{w_i\}_{i=0}^{\infty}$ for $H_0^1(\Omega)$. For each m, we define an approximate solution u_m of(4.1) as follows:

$$u_m = \sum_{i=1}^{m} g_i^m w_i \tag{3.3}$$

such that

$$\langle u'_m(t), w_i\rangle + \langle \nabla u_m(t), \nabla w_i\rangle = \langle \mu u_m(t)(1-u_m(t)), w_i\rangle \ \ a.e.\, t \in]t_0, t_1[\ i = 1, 2, ..., m \tag{3.4}$$

and

$$u_m(t_0) = b_0^{(m)} \tag{3.5}$$

where $b_0^{(m)} = \sum_{j=1}^{m} \xi_j^m w_j$ is the $L^2(\Omega)$ projection of b_0 onto the span of $\{w_1, w_2, ..., w_m\}$.
Properties of projection operators imply

$$\|b_0^{(m)}\|_{L^2(\Omega)} \leq \|b_0\|_{L^2(\Omega)} \tag{3.6}$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and $\{w_i\}$ is a basis for $H_0^1(\Omega)$, we easily deduce that

$$b_0^{(m)} \to b_0 \ in\, L^2(\Omega) \ as \ m \to \infty. \tag{3.7}$$

The system of nonlinear differential equations (3.4) and (3.5) can be rewritten as

$$\begin{aligned}&\sum_{j=1}^{m}(w_j, wi)\frac{d}{dt}g_j^{(m)}(t) + \sum_{j=1}^{m}(\nabla w_j, \nabla wi)g_j^{(m)}(t)\\ &= \mu\left\langle \sum_{j=1}^{m} g_j^{(m)}(t)w_j - \sum_{j,k=1}^{m} g_j^{(m)}(t)g_j^{(m)}(t)w_jw_k, w_i\right\rangle, \ i = 1, ..., m.\end{aligned} \tag{3.8}$$

and

$$\sum_{j=1}^{m}(w_j, w_i)g_j^{(m)}(t_0) = (b_0, w_i), \ \ i = 1, ..., m. \tag{3.9}$$

The linear independency of $\{w_i\}_{i=1}^{m}$ as functions implies that the matrix with entries (w_j, w_i) is nonsingular, so that we may use the inverse of this matrix to reduce (3.8) and (3.9) to the following standard form of a system of ODEs:

$$\frac{d}{dt}g_i^{(m)}(t) - \mu g_i^{(m)}(t) + \sum_{j=1}^{m}\alpha_{ij}g_j^{(m)}(t) + \sum_{j,k=1}^{m}\beta_{ijk}g_j^{(m)}(t)g_k^{(m)}(t) = 0, \ \ i = 1, ..., m. \tag{3.10}$$

and

$$g_i^{(m)}(t_0) = \xi_i, \;\; i = 1, ..., m. \tag{3.11}$$

for $i = 1, 2, ..., m$, where α_{ij} , β_{ijk} , $\xi_i \in \mathbb{R}$ÿ and they depend on $\{w_i\}_{i=1}^m$. From standard theories of ordinary differential equations (see, e.g., [11]), the nonlinear differential system (3.10) and (3.11) has a solution defined on a maximal right interval $[t_0, \tau^(m)]$. Or equivalently, system (3.4) and (3.5) has a solution $u_m(t)$ defined on a maximal interval
Step 2: A priori estimates for $u_m(t)$: For $t \in [t_0, \tau^(m)]$, multiplying system (3.4) by $g^(m)_i(t), \;\; i = 1, ..., m$, and adding these equations up we obtain

$$\langle u_m'(t), u_m(t)\rangle + \|\nabla u_m(t)\|^2_{L^2(\Omega)} = \langle \mu u_m(t)[1 - u_m(t)], u_m(t)\rangle. \tag{3.12}$$

Thus, applying Young's inequality to the terms on the right-hand side we have

$$\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)} \leq \mu\|u_m(t)\|^2_{L^2(\Omega)} + \mu\|u_m(t)\|^3_{L^2(\Omega)}. \tag{3.13}$$

Moreover, since $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^3(\Omega)$, we may use the interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$ and Poincare's and Young's inequalities to derive

$$\begin{aligned}
\mu\|u_m(t)\|^3_{L^3(\Omega)} \leq \mu(C\|u_m(t)\|^2_{H^{\frac{1}{2}}(\Omega)})^3 &\leq \mu(\widetilde{C}\|u_m(t)\|^{\frac{1}{2}}_{L^2(\Omega)}\|u_m(t)\|^{\frac{1}{2}}_{H_0^1(\Omega)})^3 \\
&\leq \mu(\widetilde{\widetilde{C}}\|u_m(t)\|^{\frac{3}{2}}_{L^2(\Omega)}\|\nabla u_m(t)\|^{\frac{3}{2}}_{L^2(\Omega)}) \\
&\leq C_2\|u_m(t)\|^6_{L^2(\Omega)} + \epsilon\|\nabla u_m(t)\|^2_{L^2(\Omega)} \\
&\leq C_2\|u_m(t)\|^6_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)}
\end{aligned} \tag{3.14}$$

Adding up (3.14) and then applying the resulting inequality , we obtain

$$\begin{aligned}
\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)} &\leq \mu\|u_m(t)\|^2_{L^2(\Omega)} \\
\leq C_1\|u_m(t)\|^2_{L^2(\Omega)} + C_2\|u_m(t)\|^6_{L^2(\Omega)} &+ \|\nabla u_m(t)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.15}$$

$$\Rightarrow \frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} \leq C_1\|u_m(t)\|^2_{L^2(\Omega)} + C_2\|u_m(t)\|^6_{L^2(\Omega)}. \tag{3.16}$$

Integrating (3.16) from t_0 to t, where $t \in]t_0, \tau^{(m)}[$, we are led t_o

$$\|u_m(t)\|^2_{L^2(\Omega)} - \|u_m(t_0)\|^2_{L^2(\Omega)} \leq \int_{t_0}^t [2C_1\|u_m(s)\|^2_{L^2(\Omega)} + 2C_2\|u_m(s)\|^6_{L^2(\Omega)}]ds. \tag{3.17}$$

Setting $y_m(t) = \|u_m(t)\|^2_{L^2(\Omega)}$ we have

$$y_m(t) \leq y_m(t_0) + \int_{t_0}^t g(s, y_m(s))ds \tag{3.18}$$

$g(t, y) = C_1y + C_2y^3$

It is easily verified that $g \in C([0, T] \times \mathbb{R}; \mathbb{R})$ and g is nondecreasing in y for each t.
Now, let $y(t)$ be the maximal solution of the differential equation

$$\frac{dy}{dt} = 2C_1y + 2C_2y^3 \tag{3.19}$$

with the initial value

$$y_0(t) = \|y(t_0)\|^2_{L^2(\Omega)} \tag{3.20}$$

which, according to Lemma 2.7, exists on an interval $J = [t_0, t_1]$ for some $t_1 \in]t_0, T[$ (note that system (3.19) and (3.20) and t_1 are independent of m). Inequality (3.6) implies $y_m(0) \leq \overline{y}(t_0)$. Hence by Lemma 2.9 with $a = \tau^{(m)} - t0$, we have

$$\|u_m(t)\|^2_{L^2(\Omega)} = y_m(t) \leq \overline{y}(t) \leq \max_{t\in[t_0,\tau]} \overline{y}(t) = C(t_1) \quad \forall t \in]t_0, \tau^{(m)}[\tag{3.21}$$

As $[t_0, \tau^{(m)}[$ is the maximal interval of existence for (3.18), Lemma 2.8 and (3.21) implies that $\tau^{(m)} = t_1$ and the existence interval is $[t_0, t_1]$. Hence, using (3.21) again we have

$$\sup_{t\in[t_0,\tau]} \|u_m(t)\|^2_{L^2(\Omega)} \leq C(t_1) \quad \forall t \in]t_0, t_1[\;\; m = 1, 2, \dots . \tag{3.22}$$

Thus, we have shown that $\exists t_1 \in]t_0, T]$ such that

$$\{u_m\}_{m=1}^{\infty} \; belongs\ to\ a\ bounded\ set\ of\ L^{\infty}(t_0, t_1; L^2(\Omega)). \tag{3.23}$$

Using (3.17) again, we see that relations (3.6), (3.23) yield

$$\{u_m\}_{m=1}^{\infty} \; belongs\ to\ a\ bounded\ set\ of\ L^{\infty}(t_0, t_1; H^1_0(\Omega)). \tag{3.24}$$

Step 3: Passage to limits:A priori estimates (3.23) and (3.24) allow us to draw a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) such that

$$u_m \rightharpoonup u \quad weak^* \ in \ L^{\infty}(t_0, t_1; L^2(\Omega)). \tag{3.25}$$

and

$$u_m \rightharpoonup u \quad weakly \ in \ L^2(t_0, t_1; H^1_0(\Omega)). \tag{3.26}$$

for some $u \, in L^{\infty}(t_0, t_1; L^2(\Omega)) \cap L^2(t_0, t_1; H^1_0(\Omega))$ Furthermore, Lemma 2.1
with $X_0 = H^1_0(\Omega), X = L^2(\Omega) \ and \ X_1 = H^{-1}$ implies that

$$u_m \rightharpoonup u \quad strongly \ in \ L^2(t_0, t_1; L^2(\Omega)). \tag{3.27}$$

Now, let $\varphi(t) \in C^1([t_0, t_1]; \mathbb{R})$ with $\varphi(t_1) = 0$ be given.
Multiplying (3.4) by $\varphi(t)$ and integrating by parts, we are led to

$$-(b_0^{(m)}, w_i)\varphi(t_0) - \int_{t_0}^{t_1} \langle u_m(t), \varphi'(t) w_i\rangle dt + \int_{t_0}^{t_1} \langle \nabla u_m(t), \varphi(t)\nabla w_i\rangle dt$$
$$= \int_{t_0}^{t_1} \langle \mu u_m(t)(1 - u_m(t), \varphi(t) w_i\rangle dt \tag{3.28}$$

Relations (3.25), (3.26) and (3.7) imply that as $m \to \infty$,

$$\int_{t_0}^{t_1} \langle u_m(t), \varphi'(t) w_i\rangle dt \to \int_{t_0}^{t_1} \langle u(t), \varphi'(t) w_i\rangle dt \tag{3.29}$$

$$\int_{t_0}^{t_1} \langle \nabla u_m(t), \varphi(t)\nabla w_i\rangle dt \to \int_{t_0}^{t_1} \langle \nabla u(t), \varphi(t)\nabla w_i\rangle dt \tag{3.30}$$

$$\int_{t_0}^{t_1} \langle u_m(t), \varphi(t) w_i\rangle dt \to \int_{t_0}^{t_1} \langle u(t), \varphi(t) w_i\rangle dt \tag{3.31}$$

$$(b_0^{(m)}, w_i)\varphi(t_0) \to (b_0, w_i)\varphi(t_0) \tag{3.32}$$

As for the nonlinear term, using the generalized Holder's inequality and the facts that wi is a spatial function and that $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\left| \int_{t_0}^{t_1} \langle u_m^2(t) - u^2(t), \varphi(t) w_i \rangle dt \right|$$

$$\leq \int_{t_0}^{t_1} |\varphi(t)| \int_{\Omega} |u_m(t,X) - u(t,X)|.|u_m(t,X) + u(t,X)|.|w_i(X)| dX dt$$

$$\leq \int_{t_0}^{t_1} |\varphi(t)| \|u_m(t,X) - u(t,X)\|_{L^2(\Omega)}.\|u_m(t,X) + u(t,X)\|_{L^4(\Omega)}.\|w_i\|_{L^4(\Omega)} dt$$

$$\leq C\|\varphi(t)\|_{L^\infty(t_0,t_1)} \|w_i\|_{H_0^1(\Omega)} \int_{t_0}^{t_1} \|u_m(t) - u(t)\|_{L^2(\Omega)}.\|u_m(t,X) + u(t,X)\|_{H_0^1(\Omega)}.dt$$

$$\leq C\|\varphi(t)\|_{L^\infty(t_0,t_1)} \|w_i\|_{H_0^1(\Omega)} \|u_m - u\|_{L^2(t_0,t_1,L^2(\Omega))}.\|u_m + u\|_{L^2(t_0,t_1,H_0^1(\Omega))} \tag{3.33}$$

From (3.24), (3.27) and (3.33), we easily deduce that as $m \to \infty$,

$$\int_{t_0}^{t_1} \langle u_m^2(t), \varphi(t) w_i \rangle dt \to \int_{t_0}^{t_1} \langle u^2(t), \varphi(t) w_i \rangle dt \tag{3.34}$$

Relations (3.29), (3.30) and (3.32) allow us to pass to the limits in (3.4) to obtain

$$-(b_0, w_i)\varphi(t_0) - \int_{t_0}^{t_1} \langle u(t), \varphi'(t) w_i \rangle dt + \int_{t_0}^{t_1} \langle \nabla u(t), \varphi(t) \nabla w_i \rangle dt$$
$$= \int_{t_0}^{t_1} \langle \mu u(t)(1 - u(t), \varphi(t) w_i \rangle dt \tag{3.35}$$

for each $i = 1, 2, ...$ Using the linearity in wi of(3.35) and the fact that $\{w_i\}$ is total in $H_0^1(\Omega)$ we deduce that

$$-(b_0, \phi)\varphi(t_0) - \int_{t_0}^{t_1} \langle u(t), \varphi'(t)\phi \rangle dt + \int_{t_0}^{t_1} \langle \nabla u(t), \varphi(t) \nabla \phi \rangle dt$$
$$= \int_{t_0}^{t_1} \langle \mu u(t)(1 - u(t), \varphi(t)\phi \rangle dt \quad \forall \phi \in H_0^1(\Omega) \tag{3.36}$$

In particular, (3.36) holds for all $\phi \in \mathscr{D}(0, T)$ so that u satisfies

$$\langle u'(t), \phi \rangle + \langle \nabla u(t), \nabla \phi \rangle = \langle \mu u(t)(1 - u(t), \phi \rangle dt \quad \forall \phi \in H_0^1 \tag{3.37}$$

in the sense of distributions (in time). By [17] we have that $u' \in L^2(0, T, H^{-1}(\Omega))$

Finally, it remains to prove that u satisfies the initial condition $u(t_0) = b_0$. To this end, we multiply (3.37) by $\varphi(t)$ and integrate by parts. This leads us to

$$-(u(t_0), \phi)\varphi(t_0) - \int_{t_0}^{t_1} \langle u(t), \varphi'(t)\phi \rangle dt + \int_{t_0}^{t_1} \langle \nabla u(t), \varphi(t) \nabla \phi \rangle dt$$
$$= \int_{t_0}^{t_1} \langle \mu u(t)(1 - u(t), \varphi(t)\phi \rangle dt \quad \forall \phi \in H_0^1(\Omega) \tag{3.38}$$

A comparison of(3.36) and (3.38) yields
$(b_0) - u(t_0), \phi)\varphi(t_0) = 0 \quad \forall \phi \in H_0^1(\Omega)$. Upon choosing a φ with $\varphi(t_0) = 1$, we have
$(b_0) - u(t_0), \phi) = 0 \quad \forall \phi \in H_0^1(\Omega)$.
i.e. $b_0 = u(t_0)$ a.e. Ω. This completes the proof of Theorem 3.2. □

4 Maximum principles

Let u be a local solution, i.e., a solution of (3.1) and (3.2), that was guaranteed to exist on an interval $]t_0, t_1[$ in the previous section. We are going to show that under suitable assumptions on u_0, the solution u satisfies $0 \leq u \leq 1$. . This result will also allow us to show the global existence in the next section. We first recall the well-known Gronwall Lemma:

Lemma 4.1. *(Gronwall) Assume that w, z are nonnegative, continuous functions on $[a, b]$; $K \geq 0$ is a constant; and*

$$z(t) \leq K + \int_a^t z(s)w(s)ds \quad \forall t \in [a, b]$$

Then

$$z(t) \leq K \exp\left\{ \int_a^t z(s)w(s)ds \right\} \quad \forall t \in [a, b]$$

in particular, if $K = 0$, then $z(t) \equiv 0$.

Proof. See[16]. □

Now we prove the nonnegativity of the local solution u when u_0 are nonnegative

Theorem 4.2. *Assume that $b_0 \in L^2(\Omega)$ and $b_0(x) \geq 0$ almost everywhere in Ω. If $u(t, x)$ is a solution of (3.1) and (3.2) where $t_0 \in [0, T[$ and $t_1 \in]t_0, T[$, then $u \geq 0$ almost everywhere in $]t_0, t_1[\in \Omega$.*

Proof. For almost every $t \in [t_0, t_1]$, we set $\phi(x) = u_-(t, x) = max\{-u(t, x), 0\}$ in (3.1) we have

$$\frac{d}{dt}\|u_-(t)\|^2_{L^2(\Omega)} + \|\nabla u_-(t)\|^2_{L^2(\Omega)} = \int_\Omega [u_-(t)]^2[1 - u_-(t)]dX.$$

$$\leq \mu\|u_-(t)\|^2_{L^2(\Omega)} + \mu\|u_m(t)\|^3_{L^3(\Omega)}$$

and (3.16)we have

$$\frac{d}{dt}\|u_-(t)\|^2_{L^2(\Omega)} \leq 2\mu\|u_-(t)\|^2_{L^2(\Omega)} + 2C\mu\|u_m(t)\|^6_{L^2(\Omega)}$$

Integration from t_0 to t leads us to

$$\|u_-(t)\|^2_{L^2(\Omega)} \leq \|u_-(t_0)\|^2_{L^2(\Omega)} + \int_{t_0}^t 2\mu(1 + \|u_-(s)\|^4_{L^2(\Omega)})\|u_m(s)\|^2_{L^2(\Omega)}ds \tag{4.1}$$

where $t \in [t_0, t_1]$. The identity

$$u_- = max\{-u, 0\} = \frac{|u| - u}{2}$$

and the result $u \in C([t_0, t_1]; L^2(\Omega))$ imply $u_- \in C([t_0, t_1]; L^2(\Omega))$

Setting $K = \|u_-(t_0)\|^2_{L^2(\Omega)} = 0$, $z = \|u_-(s)\|^2_{L^2(\Omega)}$ and $w = 2\mu(1 + \|u_-(s)\|^4_{L^2(\Omega)})$ we obtain from (4.1) that

$$z(t) \leq K + \int_a^t z(s)w(s)ds \quad \forall t \in [t_0, t_1]$$

Hence, Lemma 4.1 implies $z(t) \equiv 0\, on [t_0, t_1]$, i.e $\|u_-(t)\|^2_{L^2(\Omega)} \equiv 0$ which implies $u_-(t, x) = 0$ almost everywhere in $]t_0, t_1[\times\Omega$. □

Next we show that the local solution u is bounded above if the initial value is bounded above.

Theorem 4.3. *Suppose that the assumptions of Theorem 4.2 hold. Assume further that $b_0(x) \leq 1$ almost everywhere in Ω. Then the local solution $u(t,x)$ of (3.1) and (3.2) satisfies $u \leq 1$ almost everywhere in $]t_0, t_1[\times\Omega$.*

Proof. For almost every $t \in [t_0, t_1]$ we set $\phi(x) = (u-1)_+(t,x) = max\{u(t,x) - 1, 0\}$ in (4.1) and integrate in t to obtain

$$\langle u'(t), (u-1)_+\rangle + \int_\Omega \nabla u \nabla (u-1)_+ dX = \int_\Omega u(1-u)(u-1)_+ dX$$

The last equation can be written as

$$\frac{1}{2}\frac{d}{dt}\|(u-1)_+\|^2_{L^2(\Omega)} + \|\nabla(u-1)_+\|^2_{L^2(\Omega)} = -\int_\Omega u[(u-1)_+]^2 dX \leq 0$$

Hence $(u-1)_+(t,x) = 0$ almost everywhere in $]t_0, t_1[\times\Omega$. □

Remark 4.4. We point out that the proof of lamma 4.1 and theorem 4.2 do not apply to the finite dimensional Galerkin solution u_m studied in the previous section. The reason is that $[u_m]_-$ or $[u_m - 1]_+$ does not necessarily lie in the span of $\{w_1, w_2, ..., w_m\}$; as a result, we cannot set the test function to be $[u_m]_-$ or $[u_m - 1]_+$ in (3.4).

5 Global existence

From the local existence result (Theorem 4.1) and the maximum principles (Theorems 4.2 and 4.3), we concluded that under suitable assumptions on $u_0(x)$, a solution u for (3.1) and (3.2) exists on some interval $[t_0, t_1]$ and $0 \leq u \leq 1$ almost everywhere in $]t_0, t_1[\times\Omega$. Based on these we expect a global solution to exists on any finite time interval $[0, T]$. Indeed, we will prove the global existence and uniqueness in this section. While the first global existence theorem below is an easy consequence of the local existence theorem and maximum principles, the second global existence theorem requires additional effort. The following theorem gives the global existence and uniqueness of the solution.

Theorem 5.1. *Let $T \in]0, \infty[$ be given. Assume that $0 \leq u_0(x) \leq 1$ a.e. $x \in \Omega$. Then there exists a unique function $u \in W(0,T)$ such that u is a solution of (2.2) and (2.3) and*
$0 \leq u \leq 1$ a.e. in $]0, T[\times\Omega$.

Proof. By Theorem 3.2, there is a $t_1 > 0$ such that there exists a function $u \in W(0, t_1)$ as a solution of (2.2) and (2.3) on $[0, t_1]$. By Theorems 4.2 and 4.3, we also have
$0 \leq u(t,x) \leq 1$ a.e. $(t,x) \in]0, t_1[\times\Omega$.We let

$\bar{t} = sup\{\widetilde{t} : there\, exists\, a\, u \in W(0,\widetilde{t}) such\ that (3.1) and (3.2) hold for a.e.t \in]0, .t[\}$ Then $\widetilde{t}$ must equals T. For otherwise, Theorem 3.2 would allow us to continue the solution beyond $\widetilde{t}$ and this would contradict the maximality assumption of $\widetilde{t}$ (here we used the easily verifiable fact that if $u \in W(0,\widetilde{t})$ and $u \in W(\widetilde{t}, \widetilde{t} + \delta)$ for some $\delta > 0$, then $u \in W(0, \widetilde{t} + \delta)$.)

To prove the uniqueness of the solution, we suppose u_1 and u_2 are two solutions of (2.2) and (2.3) and set $v = u_1 - u_2$. Then v satisfies

$$\langle v'(t), \phi\rangle + \langle \nabla v(t), \nabla\phi\rangle = \langle \mu v(t)(1 - v(t)), \phi\rangle \ \ \forall \phi \in H^1_0(\Omega),\ a.e.\, t \in]0, T[\qquad (5.1)$$

and

$$v(0,x) = 0. \tag{5.2}$$

Setting $\phi = v(t,.)$ in (5.1) we have

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)} = \mu\|v(t)\|^2_{L^2(\Omega)} - \mu\langle v^2, v\rangle,\ a.e.\, t \in]0,T[$$

so that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)} \leq \mu\|v(t)\|^2_{L^2(\Omega)},\ a.e.\, t \in]0,T[$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} \leq \mu\|v(t)\|^2_{L^2(\Omega)},\ a.e.\, t \in]0,T[\tag{5.3}$$

Thus, using Gronwall's inequality and (5.2) we obtain

$$\|v(t)\|^2_{L^2(\Omega)} = 0,\ a.e.\, t \in]0,T[$$

From which we conclude $v = 0\ \ a.e.\, in]0,T[\times\Omega$. This completes the proof. □

Below, we prove the global existence , we first establish a new a priori estimate

Lemma 5.2. *Assume that* $0 \leq b_0(x) \leq 1$ *a.e.* $x \in \Omega$. *Let* u *be a solution of (2.2) and (2.3). Then* $t \in [0,T]$,

$$\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T,H^1_0(\Omega))} + \|u'\|^2_{L^2(0,T,H^{-1}(\Omega))} \leq \|u(t_0)\|^2_{L^2(\Omega)} + C_1 T \tag{5.4}$$

Proof. By Theorems 4.2 and 4.3, we have $0 \leq u(t,x) \leq 1$ almost everywhere in $]0,T[\in\Omega$. Thus, for a.e.$t \in [0,T]$, upon setting $\phi(x) = u(t,x)$ in (2.2) we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} = \int_\Omega \mu[u(t,X)]^2[1-u(t,X)]dX$$

$$\leq \int_\Omega \mu dX \leq \mu C_1(\mu,\Omega) \tag{5.5}$$

where $C_1 > 0$ are constants. Integration from 0 to t yields

$$\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T,H^1_0(\Omega))} \leq \|u(t_0)\|^2_{L^2(\Omega)} + C_1 T \tag{5.6}$$

From Eq. (2.2) we obtain

$$\langle u'(t),\phi\rangle + \int_\Omega \nabla u(t,X)\nabla\phi(X)dX = \int_\Omega \mu[u(t,X)]^2[1-u(t,X)]\phi(X)dX$$

$$\leq \|u(t)\|_{H^1_0(\Omega)} + \|\phi)\|_{H^1_0(\Omega)} + \mu C\int_\Omega |\phi(X)|dX$$

$$\leq \|u(t)\|_{H^1_0(\Omega)} + \|\phi)\|_{H^1_0(\Omega)} + \mu C(\mu,\Omega)\|\phi(X)\|$$

for all $\phi \in H^1_0(\Omega)$ and a.e. $t \in]0,T[$. Upon taking the supremum over $\phi \in H^1_0(\Omega)$ we obtain

$$\|u'(t)\|^2_{H^{-1}(\Omega)} \leq \|u\|^2_{H^1_0(0,T,H^1_0(\Omega))} + C(\mu,\Omega) \tag{5.7}$$

for a.e. $t \in]0,T[$. Combining (5.6) and (5.7) we obtain (5.4). □

Theorem 5.3. *Let $T \in]0, \infty[$ be given. $0 \leq u_0(x) \leq 1$ a.e. $x \in \Omega$. Then there exists a unique function $u \in W(0,T)$ such that u is a solution of (2.2) and (2.3), $0 \leq u \leq 1$ a.e. in $]0,T[\times\Omega$, and*

$$\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T,H^1_0(\Omega))} + \|u'\|^2_{L^2(0,T,H^{-1}(\Omega))} \leq \|u(t_0)\|^2_{L^2(\Omega)} + C_1 T \tag{5.8}$$

Proof. For each m, Theorem 5.1 implies the existence of a $u_m \in W(0,T)$ such that

$$\langle u'_m(t), \phi\rangle + (\nabla u_m(t), \nabla\phi) = \mu(u_m(t)(1-u_m(t)), \phi) \;\; \forall\phi \in H^1_0(\Omega), \;\; a.e.\, t \in]0,T[\tag{5.9}$$

and

$$u(0,x) = u_0. \tag{5.10}$$

Moreover, we have that $0 \leq u_n(x,t) \leq 1$ a.e. $]t,x[\in]0,T[\times\Omega$. By Lemma 5.2, we have that,

$$\|u_m\|^2_{L^2(\Omega)} + \|u'_m\|^2_{L^2(0,T,H^1_0(\Omega))} \leq \|u(t_0)\|^2_{L^2(\Omega)} + C_1 T \tag{5.11}$$

Estimate (5.11) allows us to extract a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) such that

$$u_m \rightharpoonup u \;\; weak\, in \;\; L^2(0,T;H^1_0(\Omega)). \tag{5.12}$$

$$u'_m \rightharpoonup u' \;\; weakly \;\; in \;\; L^\infty(0,T;H^{-1}(\Omega)). \tag{5.13}$$

and

$$u_m \rightharpoonup u \;\; strongly \;\; in \;\; L^2(0,T;L^2(\Omega)). \tag{5.14}$$

By repeating the arguments used in the proof of Theorem 3.2 (Step 3) we may pass to the limits in Eq. (5.9) to show

$$\langle u', \phi(t)\rangle + (\nabla u(t), \nabla\phi) = \mu(u(1-u), \phi) \;\; \forall\phi \in H^1_0(\Omega), \;\; a.e.\, t \in]0,T[\tag{5.15}$$

The estimate (5.8) follows directly from (5.11). Theorems 4.2 and 4.3 imply $0 \leq u \leq 1$ a.e. in $]0,T[\times\Omega$. Similar to the proof of Theorem 5.1, we may prove the uniqueness of the solution. This completes the proof □

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THE FULL DATA CALDERÓN'S PROBLEM IN HIGHER DIMENSIONS

AOUDJ MANAL

ABSTRACT. In this work, we study a nonlinear inverse problem for an elliptic partial differential equation known as the Calderón problem or the inverse conductivity problem. Based on several results, we briefly summarize them to motivate this research field. We give a general view of the full data problem by reviewing the available results for isotropic smooth conductivities in higher dimensions. After reducing the original problem to the inverse problem for a Schrödinger equation, we use the key ingredient of complex geometrical optics solutions to show uniqueness. After extending the ideas of the uniqueness proof result, we establish a stable dependence between the conductivity and the boundary measurements. We answer the reconstruction question by covering a useful method due to Nachman. Two open problems of significant interest are proposed to check whether it is possible to extend the above results.

KEYWORDS AND PHRASES. Calderón problem, Inverse conductivity problem, Dirichlet-to-Neumann map, Complex geometrical optics solutions, Boundary integral equation.

1. INTRODUCTION

What is the inverse conductivity problem, you may ask? Well, to answer this question, as the name of the problem indicates, we should first consider the direct conductivity problem, given by

$$\begin{cases} \nabla.\gamma\nabla w = 0 & \text{in } \Omega, \\ w = f & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and γ a positive real-valued function representing the electrical conductivity of Ω. Physically interpreted, the application of a voltage $f \in H^{1/2}(\partial\Omega)$ on the boundary induces an electrical potential w in the interior of Ω, where $w \in H^1(\Omega)$ is the unique weak solution of the elliptic boundary value problem (1).

We define the Dirichlet-to-Neumann map (DN map) Λ_γ by relating a boundary voltage f (Dirichlet data) to the flux at the boundary $\gamma\frac{\partial w}{\partial \nu}$ (Neumann data) as follows

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

$$f \mapsto \Lambda_\gamma(f) = \gamma \frac{\partial w}{\partial \nu}\bigg|_{\partial\Omega},$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative at $\partial\Omega$.

From the variational formulation of the problem (1), it follows the following

Alessandrini identity [2].

$$\langle \Lambda_\gamma f, g\rangle = \left\langle \gamma \frac{\partial w}{\partial \nu}, g \right\rangle = \int_\Omega \gamma \nabla w \nabla z dx \quad \forall f, g \in H^{1/2}(\partial\Omega),$$

where $z \in H^1(\Omega), z\big|_{\partial\Omega} = g$. It follows from this definition that Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. In this context, Calderón in his pioneer paper [9] formulated the Calderón problem as being the problem studying the inversion of the map $\gamma \mapsto \Lambda_\gamma$, i.e., the posed question is: can we determine γ from the knowledge of $\Lambda_\gamma f$ in each $f \in H^{1/2}(\partial\Omega)$? This inversion method is also called electrical impedance tomography (EIT). It is a medical imaging technology with several applications, including the detection of breast cancer and pulmonary imaging. See the review papers [7, 18] for more detailed arguments on this technique.

The determination of γ from the DN map has different aspects. In this paper, we answer the precedent question in the interior of the study domain by giving results on the three aspects of uniqueness, stability, and reconstruction. For the boundary determination: in the case of smooth conductivities Kohn and Vogelius [20] proved that Λ_γ determines γ and all its normal derivatives on the boundary. More general results were shown in [1, 32]. In particular, Brown [8] proved that we could cover the boundary values of a $W^{1,1}$, or a C^0 conductivity from the knowledge of Λ_γ.

The two-dimensional problem is also of significant interest but differs significantly from the higher dimensional one so that different techniques are used to address this case. Nachman [24] was the first who proved uniqueness for $\gamma \in W^{2,d}, d > 1$ in the plane. We refer readers to the work of Astala and Päivärinta [4] on bounded measurable conductivities for a deeper understanding of the problem in the plane.

In the following, we will only consider isotropic conductivities, which are not dependent on direction. When γ depends on direction, we are in the presence of the anisotropic Calderón problem. In the plane, uniqueness was shown for L^∞ anisotropic conductivities in [3]. For $n \geq 3$, this problem is also called Calderón's inverse problem on Riemannian manifolds, and as Lassas and Uhlmann pointed out in [22], this is a geometrical problem that up to now remains open.

While the current paper deals only with the full data problem, we note that the partial data problem is subject to huge advances. The partial data type problem aims to reduce as much as possible the part of the boundary where measurements are taken and where excitations on the studied body are imposed because, from a realistic point of view, it is not practical to consider measurements on the whole boundary of some domain. We refer the reader to the excellent survey paper [19] by Kenig and Salo on the recent progress in this problem.

There are several problems related to the main one. The fractional Calderón problem is a nonlocal version of the classical one. It was introduced firstly in [14]. In the present work, it is a question to study a Schrödinger operator containing an electrical potential, but if there is also a nonzero magnetic potential, we are in the presence of another variant of the standard problem, namely the Calderón problem for the magnetic Schrödinger operator [21]. By combining the two precedent problems, we can also define another closely related one, which is the inverse conductivity problem for the fractional magnetic operator, and it is the subject of [23].

As the research field on the Calderón problem is very broad, we focus our attention on its main aspects, and we refer the reader to the review works [11, 5, 34, 18] on the general problem. We propose a simplified review of the notes of Mikko Salo [35] and some chapters from [12] for the full data problem. Even though those notes are very good, they are limited by pedagogical requirements, and as this work is not bounded by these limitations, we state reconstruction results also. The rest of this article is organized in the following way. The applied notation and background knowledge are summarized in Section 2. In Section 3, we review the known uniqueness results for the full data problem for C^2 conductivities. The stability issues are reviewed in Section 4. In Section 5, we present Nachman's reconstruction method. Section 6 contains some perspectives and open problems.

2. Preliminaries

Throughout this article

- Ω denotes a bounded open set of $\mathbb{R}^n$ with boundary $\partial\Omega$.
- $n \geq 3$ denotes the space dimension.
- $q : \Omega \to \mathbb{R}$ denotes an electrical potential.
- dS denotes the surface on $\partial\Omega$.
- $\mathcal{S}(\mathbb{R}^n)$ denotes Schwartz space.
- $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.
- $D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}$ denotes the gradient of a periodic function, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$.
- $\langle , \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.
- $B_R(0)$ denotes the closed ball with center 0 and radius $R > 0$.
- $a \lesssim b$ denotes that it exists a constant $c > 0$ such that $a \leq cb$.

2.1. **Fourier transform and function spaces.** For $\xi \in \mathbb{R}^n$, the applied notation for the Fourier transform is

$$\hat{w}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi . x} w(x) dx.$$

The inverse Fourier transform is noted by

$$\check{w}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi . x} w(\xi) d\xi.$$

For $s \in \mathbb{R}$, we define Sobolev spaces $H^s(\mathbb{R}^n)$ via Fourier transform as follows

$$H^s(\mathbb{R}^n) = \{w \in \ \mathcal{S}'(\mathbb{R}^n) : \ \ \langle \xi \rangle^s \widehat{w} \in L^2(\mathbb{R}^n)\},$$

where $\langle \xi \rangle = (|\xi|^2 + 1)^{1/2}$. The associated norm is $\|w\|_{H^s(\mathbb{R}^n)} = \|\langle \xi \rangle^s \widehat{w}\|_{L^2(\mathbb{R}^n)}$. We give the following properties, which will be needed later in Section 4.

Proposition 2.1. *(Sobolev embedding) If* $w \in H^{s+k}(\mathbb{R}^n), s > n/2, k \in \mathbb{N}$, *then* $w \in C^k(\mathbb{R}^n)$ *and*

$$\|w\|_{C^k(\mathbb{R}^n)} \leq c\|w\|_{H^{s+k}(\mathbb{R}^n)}.$$

Proposition 2.2. *(Multiplication by functions) If $w \in H^s(\mathbb{R}^n), s \geq 0, f \in C^k(\mathbb{R}^n), k \in \mathbb{N}, k \geq s$, then $fw \in H^s(\mathbb{R}^n)$ and*

$$\|fw\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{C^k(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)}.$$

Proposition 2.3. *(Logarithmic convexity) If $0 \leq a \leq b,\ 0 \leq \tau \leq 1$, then*

$$\|w\|_{H^c(\mathbb{R}^n)} \leq \|w\|^{1-\tau}_{H^a(\mathbb{R}^n)} \|w\|^{\tau}_{H^b(\mathbb{R}^n)},$$

where $c = (1-\tau)a + \tau b$.

As we will see in the next section, the problem (1) can be reduced to another one for the Schrödinger equation. Then, by substituting with $w(x) = e^{i\zeta . x}(1+r)$, in this last problem, we deduce an equivalent equation for r, precisely

$$\triangle_\zeta r = (\Delta + 2i\zeta.\nabla) r = q(1+r) \ \text{ in } \Omega.$$

The right inverse of the differential operator $\triangle_\zeta$ is defined by

$$\widehat{\triangle_\zeta^{-1} f(\xi)} = p_\zeta(\xi)^{-1} \hat{f}(\xi). \tag{2}$$

with symbol

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta.\xi.$$

Using this symbol, we can define the space $\dot{X}^b_\zeta$ with the associated norm

$$||w||_{\dot{X}^b_\zeta} = |||p_\zeta(\xi)|^b \hat{w}(\xi)||_{L^2}.$$

and the inhomogeneous spaces X^b_ζ with the associated norm

$$||w||_{X^b_\zeta} = ||(|\zeta| + |p_\zeta(\xi)|)^b \hat{w}(\xi)||_{L^2}.$$

In Section 6, we will only need to use the exponent $b = \pm 1/2$. Notice that those two spaces were firstly considered by Haberman and Tataru in [16] in the spirit of Bourgain's spaces, see [6, 33].

2.2. **Faddeev's Green's function and layer operator.** While the equation (2) implicitly gives the right inverse G_ζ of $\triangle_\zeta$, the following explicit functions

$$g_\zeta(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\xi.x}}{p_\zeta(\xi)} d\xi, \qquad G_\zeta(x) = e^{i\zeta.x} g_\zeta(x), \tag{3}$$

are the Faddeev's Green's functions for $(\Delta + 2i\zeta.\nabla)$ and the Laplacian, respectively. Now, let us introduce some useful facts, which will be used later in Section 5.

Definition 2.4. *A single layer potential is an integral operator mapping functions on $\partial\Omega$ to functions in $\mathbb{R}^n$.*

Remark 2.5. *A harmonic function satisfying certain decay conditions on the exterior domain can be represented in terms of a single layer potential.*

Using the family G_ζ of Green's functions for $x \in \mathbb{R}^n \backslash \partial\Omega$, we define analogs of the classical single and double layer potentials as follows.
Single layer potential:

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x,y) f(y) dS(y). \tag{4}$$

Double layer potential:

$$D_\zeta f(x) = \int_{\partial\Omega} \frac{\partial G_\zeta(x,y)}{\partial \nu(y)} f(y) dS(y).$$

We define also for $x \in \partial\Omega$, the boundary double layer potential:

$$B_\zeta f(x) = p.v. \int_{\partial\Omega} \frac{\partial G_\zeta(x,y)}{\partial \nu(y)} f(y) dS(y). \tag{5}$$

3. Uniqueness

In this section, to answer the uniqueness question, we should ask if it is possible to determine γ from the knowledge of Λ_γ, i.e., is the map $\gamma \mapsto \Lambda_\gamma$ injective? In 1980, Alberto Calderón, who proposed the problem, gave a positive answer. He proved in his pioneer paper [9] that for γ a perturbation of the identity, the injectivity of the linearized inverse problem holds. Here, we consider the result of Sylvester and Uhlmann [31], which states the unique recovery of γ from Λ_γ. For $n \geq 3$, Sylvester and Uhlmann were the first to show uniqueness for C^2 conductivities. Their idea was to look for special solutions, which are asymptotically exponential, satisfying a certain Schrödinger equation.

Theorem 3.1. *For $j = 1, 2$, let $\gamma_j \in C^2(\bar{\Omega})$ be two positive functions. Then we have*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2 \quad in\ \Omega.$$

This Theorem can be reduced to the following one for a Schrödinger equation.

Theorem 3.2. *For $j = 1, 2$, let $q_j \in \mathcal{Q}$. Then we have*

$$\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2 \quad in\ \Omega.$$

Next, we proceed to the reduction of Theorem 3.1 to Theorem 3.2.

3.1. **Reduction of the conductivity equation to the Schrödinger equation.**

By providing a certain amount of smoothness (in our case γ has two derivatives), we can reduce the Calderón problem to the inverse boundary value problem for a Schrödinger equation. This reduction is based on the well-known Liouville transformation: if z is a weak solution of the conductivity equation $\nabla.\gamma\nabla z = 0$, then $w = \gamma^{1/2} z$ is a solution to the Schrödinger equation $(-\Delta + q)w = 0$, where the potential $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. For $q \in L^\infty(\Omega)$, and for all $f \in H^{1/2}(\partial\Omega)$, we give the following boundary value problem for the Schrödinger equation

$$\begin{cases} -\Delta w + qw = 0 & \text{in } \Omega, \\ w = f & \text{on } \partial\Omega. \end{cases} \tag{6}$$

From now on, we consider the standard assumption that 0 is not a Dirichlet eigenvalue for the Schrödinger equation. Under this condition, the problem

(6) is well-posed in the sense that it admits a unique solution. Moreover, we define $\mathcal{Q}$ as being the subset of all potentials $q \in L^\infty(\Omega)$ such that 0 is not a Dirichlet eigenvalue for $(-\Delta + q)w = 0$.
For all $q \in \mathcal{Q}$, we define the DN map Λ_q associated with (6) by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$f \mapsto \Lambda_q(f) = \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega}.$$

From the variational formulation, it is clear that

$$\langle \Lambda_q f, g\rangle = \int_\Omega (qwz + \nabla w.\nabla z)\ dx \quad \forall f, g \in H^{1/2}(\partial\Omega). \tag{7}$$

Which implies that Λ_q is a self-adjoint bounded linear mapping from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. Since $q \in \mathcal{Q}$, we can give another useful identity when $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. It is clear that the DN map Λ_q can be obtained from the DN map Λ_γ. The explicit expression relating those two maps is given by

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2} f) + \frac{1}{2}\gamma^{-1}\left.\frac{\partial \gamma}{\partial \nu} f\right|_{\partial\Omega}. \tag{8}$$

The next Corollary shows that Theorem 3.2 implies Theorem 3.1.

Corollary 3.3. *If the conditions of Theorem 3.1 are satisfied and Theorem 3.2 holds, then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2 \quad in\ \Omega.$$

Proof. Under the hypothesis of Theorem 3.1, suppose that for $j = 1, 2$, $q_j = \gamma_j^{-1/2}\Delta\gamma_j^{1/2}$, then $q_j \in \mathcal{Q}$. By applying Theorem 1.3 (Uniqueness at the Boundary) from [1], it follows that $\gamma_1 = \gamma_2$ and $\frac{\partial\gamma_1}{\partial\nu} = \frac{\partial\gamma_2}{\partial\nu}$ on $\partial\Omega$.
By using relation (8) with q_2 for all $f \in H^{1/2}(\partial\Omega)$, and replacing with the boundary values of γ and its normal derivative, it follows that $\Lambda_{q_1} f = \Lambda_{q_2} f\ \forall f \in H^{1/2}(\partial\Omega)$. Therefore, by Theorem 3.2 we deduce that $q_1 = q_2$ in the whole domain Ω.
Since $q \in \mathcal{Q}$, $(-\Delta + q)w = 0$ has a unique solution in Ω. In particular, we substitute with $q = \gamma_1^{-1/2}\Delta\gamma_1 = \gamma_2^{-1/2}\Delta\gamma_2$. Which implies that both of $\gamma_1^{1/2}$ and $\gamma_2^{1/2}$ solve the precedent Schrödinger equation with the same boundary value (by using the precedent boundary identification). It follows from the uniqueness that $\gamma_1^{1/2} = \gamma_2^{1/2}$. □

Remark 3.4. *For later use in the next subsection, we follow another direction to prove that* $\gamma_1^{-1/2}\Delta\gamma_1 = \gamma_2^{-1/2}\Delta\gamma_2 \Rightarrow \gamma_1 = \gamma_2$.
By using the identification $\triangle(\log\gamma_j^{1/2}) = \gamma_j^{-1/2}\Delta\gamma_j - |\nabla(\log\gamma_j^{1/2})|^2$*, the equation* $q_1 = q_2$ *may be written as* $\triangle z + \nabla\theta.\nabla z = 0$*, which is a linear equation for* $z = \log\gamma_1^{1/2}\gamma_2^{-1/2} \in C^2(\Omega)$*, with* $\theta = \log(\gamma_1\gamma_2)^{1/2}$*.*
By the identity $\nabla.(e^\theta\nabla z) = e^\theta(\Delta z + \nabla\theta.\nabla z)$*, and the boundary identification* $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$*, we see that* z *is a solution to the well-posed Dirichlet problem*

$$\begin{cases} \nabla.((\gamma_1\gamma_2)^{1/2}\nabla z) = 0 & in\ \Omega, \\ z = 0 & on\ \partial\Omega. \end{cases}$$

Then, z is identically zero in the whole domain Ω, and we get the aimed conclusion $\gamma_1 = \gamma_2$ in Ω.

From now on, our attention will be focused on Theorem 3.2.

3.2. **Construction of special solutions.** Next, we use Fourier analysis to construct special solutions for the Schrödinger equation.

Let $A = \{\zeta \in \mathbb{C}^n : \ \zeta.\zeta = 0\}$. If $w = e^{i\zeta.x}, \zeta \in A$, then w is a harmonic function, which solves $(-\Delta + q)w = 0$ with $q = 0$, but if $q \neq 0$, $w = e^{i\zeta.x}$ can not be a solution to the precedent equation anymore. However, we can still find solutions that look like the previous ones. This idea is due to Sylvester and Uhlmann [31].
Here we look for special solutions $w(x, \zeta), \zeta \in A$, to the equation $(-\Delta + q)w = 0$, which are asymptotically exponential, i.e., $w \sim e^{i\zeta.x}$ for $|\zeta| \to \infty$. This last asymptotic property asserts that the corrector function $r(x) = \ell(x) - 1$, with $\ell = e^{-i\zeta.x}w$ decays to zero when $|\zeta| \to \infty$. Therefore, we write

$$w(x) = e^{i\zeta.x}(1 + r), \tag{9}$$

with $r \in H^1(\Omega)$ is just a correction term that is needed to transit from an approximate solution to the exact one by taking $|\zeta| \to \infty$.
We substitute with (9) in $(-\Delta + q)w = 0$. By using the fact that $-\Delta = D^2$[1], we obtain

$$(D^2 + 2\zeta.D + q)r = -q \text{ in } \Omega. \tag{10}$$

Which shows that (9) is a solution to $(-\Delta + q)w = 0$ if and only if (10) holds. The functions $w(x) = e^{i\zeta.x}(1 + r)$ are called complex geometrical optics solutions (CGOs). To approximate them to the exact solutions, we should establish certain asymptotic bounds on the corrector term r when $|\zeta| \to \infty$.
Firstly, we give the following basic estimate for $q = 0$.

Proposition 3.5. *There exists a constant $M(n, \Omega)$ such that for any $\zeta \in A, |\zeta| \geq 1$, and $f \in L^2(\Omega)$, the function $r \in H^1(\Omega)$ solves the equation*

$$(D^2 + 2\zeta.D)r = f \text{ in } \Omega, \tag{11}$$

with the following estimates

$$\|r\|_{L^2(\Omega)} \leq \frac{M}{|\zeta|} \|f\|_{L^2(\Omega)},$$

$$\|\nabla r\|_{L^2(\Omega)} \leq M \|f\|_{L^2(\Omega)}.$$

Proof. The idea of the proof is to apply Fourier transform to (11) since it is a linear equation with constant coefficients. If we directly do that, we get

$$(\xi^2 + 2\zeta.\xi)\hat{r}(\xi) = \hat{f}(\xi),$$

which is an expression of r with a vanishing denominator. For instance, for $\xi = 0$.

[1] Since $Dw = \Sigma_{j=1}^n \frac{1}{i} \frac{\partial w}{\partial x_j}$, then $D^2 w = (\frac{1}{i})^2 \Sigma_{j=1}^n \frac{\partial^2 w}{\partial x_j^2} = -\Delta w$.

To simplify, we extend f to be zero in the cube $Q = [-\pi, \pi]^n$ outside the domain Ω. Let $\zeta = h(a_1 + a_2)$, where $h = \frac{|\zeta|}{2^{1/2}}$, and $a_1, a_2 \in \mathbb{R}^n$ are two orthogonal unit vectors that we identify with the vectors of the canonic base e_1 and e_2. Clearly, (11) becomes

$$(D^2 + 2h(D_1 + iD_2))r = f \ in \ Q.$$

We write $\omega_k = e^{i(k+1/2e_2).x}, k \in \mathbb{Z}^n$, where $\{\omega_k\}$ is an orthonormal complete set of $L^2(Q)$. By applying the theory of Hilbert spaces, we are using the smart technique of expressing the second member and the corrector term of the last equation as Fourier series on Q in slightly shifted lattices from $\mathbb{Z}^n$ to $\mathbb{Z}^n + 1/2e_2$. After some calculus, we get

$$\|r\|_{L^2(Q} \leq \frac{1}{h}\|f\|_{L^2(Q)} \text{ and } \|\nabla r\|_{L^2(Q)} \leq 4\|f\|_{L^2(Q)}.$$

This means that under the condition $|\zeta| \geq 1$, both r and ∇r are in L^2. Thus, $r \in H^1(\Omega)$ and satisfies the above two estimations. □

If $q \neq 0$, we are no more in the presence of a linear equation with constant coefficients, then the previous method is inapplicable. Now, under the condition $|\zeta| \geq \max(1, M\|q\|_{L^\infty})$, and by analogy to Proposition 3.5, we can show that the function $r \in H^1$ solving the equation

$$(D^2 + 2\zeta.D + q)r = f \text{ in } \Omega, \tag{12}$$

satisfies the same precedent L^2 norm estimates for the corrector term r and its gradient ∇r.
To prove this, we define the solution operator of equation (10) G_ζ from $L^2(\Omega)$ to $H^1(\Omega)$ by $G_\zeta(f) = r$.
We know that since $q \neq 0$, the solution of (12) may be given by $G_\zeta \tilde{f} = r \ \forall \tilde{f} \in L^2(\Omega)$, which implies that $(I + qG_\zeta)\tilde{f} = f$.
Since $|\zeta| \geq \max(1, M\|q\|_{L^\infty})$, it follows that

$$\|qG_\zeta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{M\|q\|_{L^\infty}}{|\zeta|} \leq \frac{1}{2}.$$

To conclude, we can get the desired estimates by inverting the operator $I + qG_\zeta$.

Next, we consider the existence result of CGOs from [31].

Theorem 3.6. *Let $q \in L^\infty(\Omega)$. There exists a constant $M(n, \Omega)$ such that for any $\zeta \in A, |\zeta| \geq \max(1, M\|q\|_{L^\infty})$, $\lambda \in H^2(\Omega)$, $\zeta.\nabla\lambda = 0$ in Ω. The equation $(-\Delta + q)w = 0$ has a solution $w(x) = e^{i\zeta.x}(\lambda + r)$, with $r \in H^1(\Omega)$ satisfying*

$$\|r\|_{L^2(\Omega)} \leq \frac{M}{|\zeta|}\|(-\Delta + q)\lambda\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq M\|(-\Delta + q)\lambda\|_{L^2(\Omega)}.$$

This Theorem guarantees the existence of CGOs for the Schrödinger equation, but what about the uniqueness question for the Calderón problem, since it is the subject of this uniqueness subsection? The answer will be given in the rest of the present subsection.

3.3. **Uniqueness proof.** From Corollary 3.3, we know that Theorem 3.2 implies Theorem 3.1, then it is sufficient to prove the uniqueness result for Theorem 3.2. To do that, we still need an integral identity that relates boundary measurements with interior potentials. From (7), it follows that

$$\left\langle (\Lambda_{q_1} - \Lambda_{q_2}) w_1\big|_{\partial\Omega}, w_2\big|_{\partial\Omega} \right\rangle = \int_\Omega (q_1 - q_2) w_1 w_2 \, dx, \tag{13}$$

for $q_j \in \mathcal{Q}$, and $w_j \in H^1$ uniquely solve $-\Delta w_j + q_j w_j = 0$, for $j = 1, 2$. Now, we can give the outline of the proof of Theorem 3.2.

Proof of Theorem 3.2. Since $\Lambda_{q_1} = \Lambda_{q_2}$, the integral identity (13) can be simplified to

$$\int_\Omega (q_1 - q_2) w_1 w_2 \, dx = 0. \tag{14}$$

The idea of the proof is to look for an approximation of $e^{i\zeta.x}$ by the products $w_1 w_2$. That is possible by following this reasoning: Fix $\xi \in \mathbb{R}^n$, since $n \geq 3$, we introduce two unit vectors $\omega_1, \omega_2 \in \mathbb{R}^n$ such that the set $\{\omega_1, \omega_2, \xi\}$ is orthogonal.

Let $\zeta = h(\omega_1 + i\omega_2)$, then $\zeta \in A$. If h is sufficiently large, the application of Theorem 3.6 guarantees the existence of two CGOs for $(-\triangle + q_j)w_j = 0, j = 1, 2$:

$$w_1 = e^{i\zeta.x}(e^{ix.\xi} + r_1) \text{ and } \; w_2 = e^{-i\zeta.x}(1 + r_2).$$

Moreover, $\|r_j\|_{L^2(\Omega)} < C/h$. By taking the limits as $h \to \infty$ in (14), the correction terms in w_1 and w_2 will vanish. Consequently, as mentioned before the CGOs will look like the complex exponential $e^{i\xi.x}$. Thus, the precedent integral identity (14) becomes

$$\int_\Omega (q_1 - q_2) e^{i\xi.x} \, dx = 0.$$

Since this identity holds for every $\xi \in \mathbb{R}^n$, we extend by zero the function $(q_1 - q_2)(\xi)$ on $\mathbb{R}^n \backslash \Omega$. Then, by the uniqueness Theorem of the Fourier transform [30], we deduce that $q_1 = q_2$ in Ω. This means that the map $q \mapsto \Lambda_q$ is one-to-one. Thus, the inverse problem for the Schrödinger equation has a unique solution. Therefore, the uniqueness of the inverse conductivity problem holds (Theorem 3.2). □

4. Stability

By definition: the stability of a problem is that the behavior of the solution changes continuously with the change of initial conditions. This means that a small change in data leads to a small change in the solution. Thus, this section aims to establish the estimation

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \varpi \left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)} \right).$$

We observe that the difference of the conductivities is taken in the L^∞ norm. Thanks to an example given by Alessandrini [2], if we take $\gamma_j \in L^\infty(\Omega), j = 1, 2$, the precedent estimate is invalid. Logical reasoning then is to impose some a priori constraints on the conductivities γ_j as it is announced in the following result of Alessandrini [2].

Theorem 4.1. *For $j=1,2$, let $\gamma_j \in H^{s+2}(\Omega), s \geq n/2$ be two positive functions satisfying $1/N \leq \gamma_j \leq N$, and $\|\gamma_j\|_{H^{s+2}} \leq N$. There exist constants $t(s,n) \in (0,1)$, and $C(n,\Omega,N,s) > 0$ such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \varpi\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)}\right),$$

where ϖ is a modulus of continuity satisfying $\varpi(\tau) \leq C \mid \log \tau \mid^{-t}$, $0 < \tau < 1/e$.

Remark 4.2. *Since $\gamma_j \in H^{s+2}(\Omega)$ for $s \geq n/2$, then $\gamma_j \in C^2(\overline{\Omega})$ by Proposition 2.1.*

Boundary stability: Under the same assumptions of Theorem 4.1, we have the following result for stability at the boundary.

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)}\right). \tag{15}$$

We also give the stability result for the Schrödinger equation.

Theorem 4.3. *For $j=1,2$, let $q_j \in \mathcal{Q}$, with $\|q_j\|_{L^\infty(\Omega)} \leq N$. There exists a constant $C(n,\Omega,N,s) > 0$ such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq \varpi\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)}\right),$$

where ϖ satisfies $\varpi(\tau) \leq C \mid \log \tau \mid^{-\frac{2}{n+2}}$, $0 < \tau < 1/e$.

To prove this, we consider $\xi \in \mathbb{R}^n$, and we define the set B by

$$B = \{\zeta_j \in \mathbb{C}^n : \ \zeta_j.\zeta_j = 0, \ |\zeta_1| = |\zeta_2| = h, \ \zeta_1 + \zeta_2 = \xi, j = 1,2\}. \tag{16}$$

The application of Theorem 3.6 under the condition $h \geq \max(1, MN)$ ensures the existence of CGOs:

$$w_1 = e^{i\zeta_1.x}(1 + r_1) \text{ and } w_2 = e^{i\zeta_2.x}(1 + r_2),$$

for $(-\Delta + q_j)w_j = 0$, with $\|r_j\|_{L^2(\Omega)} \leq \frac{M}{h}\|q_j\|_{L^\infty(\Omega)}$, and M depending only on n and Ω.

We note by $\widetilde{q_j}$ the extension of q_j by zero to $\mathbb{R}^n$, by using the previous integral identity (13) with some calculus (considering $\Omega \subseteq B_R(0)$ and $h \leq e^{Rh}$), we get

$$|\widehat{(\widetilde{q_1} - \widetilde{q_2})}| \leq C\left(e^{4Rh}\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)} + 1/h\right).$$

This last estimation means that the Fourier transform of the extension of $q_1 - q_2$ to $\mathbb{R}^n$ is a L^1 function.

For a constant $\rho > 0$, from the definition of the norm in $H^{-1}(\mathbb{R}^n)$ it is clear that

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq \|q_1 - q_2\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{|\xi| \leq \rho} \left|\langle\xi\rangle^{-1}\widehat{(q_1 - q_2)}\right|^2 d\xi + \int_{|\xi| > \rho} \left|\langle\xi\rangle^{-1}\widehat{(q_1 - q_2)}\right|^2 d\xi.$$

Then, with an appropriate choice of h and ρ we get the desired conclusion.

Now, to reduce the stability result for the conductivity equation to the one for the Schrödinger equation, we need more facts about Sobolev spaces.

Then, we recall properties 2.1, 2.2, and 2.3 from Section 2. Since we are working on a bounded domain with a smooth boundary, we need to use the corresponding of the previous properties on Ω and $\partial\Omega$. This is possible by introducing the extension operator and via $H^s(\mathbb{R}^{n-1})$, respectively. Notice that the continuous functions are defined on $\bar{\Omega}$, and the condition on s becomes $s \geq \frac{n-1}{2}$ on $\partial\Omega$ since that $\partial\Omega \subset \mathbb{R}^{n-1}$.
Moreover, we need the following inequality that relates the difference of the DN maps for conductivities to the one for potentials.

Lemma 4.4. *Under the same conditions of Theorem 4.1, there exists a constant $C(n, \Omega, N, s)$ such that*

$$\begin{aligned}(17)\quad & \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\ & \leq C\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\frac{2}{2s+3}}_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right).\end{aligned}$$

Proof. For $f \in H^{1/2}(\partial\Omega)$, we use equation (8) to calculate $(\Lambda_{q_1} - \Lambda_{q_2})f$. We estimate the $H^{-1/2}$ norm of the resulting expression by the triangle inequality, then we use the a priori conditions and Proposition 2.1 to get

$$\begin{aligned}(18)\quad & \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\ & \leq C\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)}\right).\end{aligned}$$

By Proposition 2.1 again, Proposition 2.3, the trace Theorem, and the estimates on γ_j, we can change the C^1 norm in (18) to the L^∞ norm.
The estimate (17) follows then from the stability at the boundary result (15). □

To conclude this section, we give the idea of the proof of Theorem 4.1. We recall from Remark 3.4, the function $z = \log \gamma_1^{1/2} \gamma_2^{-1/2} \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases} \nabla.(\gamma_1\gamma_2)^{1/2}\nabla z = (\gamma_1\gamma_2)^{1/2}(q_1 - q_2) & \text{in } \Omega, \\ z = \frac{1}{2}\log\gamma_1 - \frac{1}{2}\log\gamma_2 & \text{on } \partial\Omega. \end{cases}$$

By applying Theorem 4.3 and Lemma 4.4, we can obtain a bound for z in $H^1(\Omega)$ in terms of z in $H^{1/2}(\partial\Omega)$. Similarly to the precedent proof, we change those norms to the L^∞ norm. Then, by the a priori constraints, Proposition 2.1,2.3, and estimate (15), we can deduce the continuous dependence of the initial data with the solution. Thus, the stability for the Calderón problem.

5. Reconstruction

In the present section, we consider the Calderón problem of reconstructing a conductivity from measurements on the boundary, and we aim to offer the interested reader a short introduction to this aspect. Once uniqueness holds, we can interest in the reconstruction problem.

In 1988, for three and higher dimensions, Nachman [25] was the first who provided a constructive procedure to recover $\gamma \in C^{1,1}$ from the knowledge of Λ_γ. In practice, Nachman's reconstruction procedure was widely applied in the implementation of algorithms [29].

For a constant $c_0 > 0$, we consider the condition

$$\gamma(x) \geq c_0. \tag{19}$$

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with a $C^{1,1}$ boundary, and let $\gamma \in C^{1,1}(\bar{\Omega})$ satisfying (19). Then there is a procedure to reconstruct γ uniquely from Λ_γ.*

Novikov [27] has independently show a similar result to the previous one given by Nachman. He was the first who introduced the key ingredient of the boundary integral equation, which will be explained later in this section.

5.1. **Nachman's methode.** Now, we briefly review the outlines of the proof of Theorem 5.1, and the main theoretical tools used therein.

5.1.1. *Preliminary reductions.* To simplify the problem, we use the following two types of reductions. On one hand, we know from the precedent sections that the conductivity problem could be reduced to the Schrödinger problem by a well-known transformation under the condition that the conductivities are sufficiently regular (which is the case here). On the other hand, Nachman proceed to a reduction of γ in a neighborhood of $\partial\Omega$. His idea was to reduce the Calderón problem to a problem having a constant $\gamma \equiv 1$ near $\partial\Omega$, then to extend γ outside the study domain Ω such that the initial regularity assumption is conserved. Thus, solving the extended problem on the large domain means that the original problem on Ω is implicitly solved. The main idea behind this reduction is based on the indispensable step of reconstructing the boundary value of the unknown conductivity and its derivative from the DN map, also called reconstruction at the boundary. From identity (8), it is clear that to find the value of Λ_q, we need a procedure to recover the values of γ and $\frac{\partial\gamma}{\partial\nu}$ on the boundary $\partial\Omega$ from Λ_γ. Thus, we deduce the importance of boundary reconstruction, which depends on the regularity of both the domain boundary and the conductivity itself. Under the same assumptions of Theorem 5.1, Nachman ([25], Section 4) yields to an appropriate boundary determination. The desired conclusion behind those reductions is to possed a potential q having a compact support in Ω.

5.1.2. *Boundary integral equation.* Here, we will more carefully describe each step in the reconstruction procedure in higher dimensions. For $n \geq 3$, the valuable tool of CGOs, which was viewed above to show the uniqueness in Calderón problem in the work of Sylvester and Uhlmann [31], was used later by Nachman in Theorem 5.1 and by Novikov in [27] independently to reconstruct the conductivity γ.

We will describe Nachman's idea [25] as follows. As it was explained above, we can give the boundary reconstruction of γ and $\frac{\partial\gamma}{\partial\nu}$ from the DN map. Then, if Λ_γ is known, Λ_q can be calculated from identity (8). Hence, the problem is reduced to the reconstruction of q from Λ_q. Once we have the value of $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, we solve the following problem to deduce γ.

$$\begin{cases} -\Delta w + qw = 0 & \text{in } \Omega, \\ w = \gamma^{1/2} & \text{on } \partial\Omega. \end{cases}$$

Now, let $q_1 = q$, $q_2 = 0$ in the integral identity (13). Then we get

$$\int_\Omega q w_1 w_2 \, dx = \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(w_1|_{\partial\Omega}) w_2|_{\partial\Omega} \, dS, \tag{20}$$

where $w_1, w_2 \in H^1(\Omega)$ solves $-\Delta w_1 + q w_1 = 0$, and $-\Delta w_2 = 0$, respectively. In the following we use expression (20) and appropriate CGOs to reconstruct the Fourier transform of q. We consider $\xi \in \mathbb{R}^n, \xi \neq 0$, and $\zeta_1, \zeta_2 \in B$, where the set B is defined in (16). The application of the argument from Section 3 ensures the existence of CGOs $w_1 = e^{i\zeta_1 . x}(1 + r)$ for $-\Delta w_1 + q w_1 = 0$, with the correction term r decaying to zero when $|\zeta_1| \to \infty$. Furthermore, the appropriate choice of $\zeta_2 . \zeta_2 = 0$ implies that $\Delta e^{i\zeta_2 . x} = 0$.
By substituting in (20) and by using the decay property of r, we have

$$\hat{q}(\xi) = \lim_{h \to \infty} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(w_1|_{\partial\Omega}) e^{i\zeta_2 . x}|_{\partial\Omega} \, dS. \tag{21}$$

From (21), we deduce that the Fourier transform of q for $\xi \neq 0$ can be recovered from the DN map if $w_1|_{\partial\Omega}$ is known. We know that q is compactly supported, then $\hat{q}(\xi)$ is countinuous so that $\hat{q}(0)$ can be determined by continuity [30]. Hence, $\hat{q}(\xi)$ is known as a tempered distribution, and the potential q can be recovered in $\mathbb{R}^n$ by simply inverting the Fourier transform. Therefore, it is a question to get the value of $w_1|_{\partial\Omega}$ to recover $\hat{q}(\xi)$.

The aim now is to find a method to calculate $w_1|_{\partial\Omega}$. The idea is to look at the exterior problem, which means that we extend q to $\mathbb{R}^n$ to be $q = 0$ outside the study domain Ω. Since $q = 0$ in $\mathbb{R}^n \backslash \bar{\Omega}$, the equation $(-\Delta + q) w_1|_{\partial\Omega} = 0$ in $\mathbb{R}^n$ becomes $-\Delta w_1|_{\partial\Omega} = 0$ in $\mathbb{R}^n \backslash \bar{\Omega}$. Therefore, the function w_1 is a solution to the following exterior problem.

$$\begin{cases} -\Delta w_1 = 0 \text{ in } \mathbb{R}^n \backslash \bar{\Omega}, \\ w_1|_{\partial\Omega} = f_\zeta, \\ \frac{\partial w_1}{\partial \nu}|_{\partial\Omega} = \Lambda_q f_\zeta. \end{cases} \tag{22}$$

For a fixed $R > R_0$ such that $\Omega \subset B_R(0)$, it is known from [25] that if w_1 satisfies the following analog of Sommerfeld radiation condition

$$\lim_{R \to \infty} \int_{|y|=R} \left(G_\zeta(x,y) \frac{\partial (w_1 - e^{i\zeta . x})}{\partial \nu(y)} - (w_1 - e^{i\zeta . x}) \frac{\partial G_\zeta(x,y)}{\partial \nu(y)} \right) dS(y) = 0, \tag{23}$$

then by using Green's formula in (22), we can show that the boundary value $w_1|_{\partial\Omega}$ can be characterized as the unique solution f_ζ of the following boundary integral equation of Fredholm type.

$$e^{i\zeta . x} - S_\zeta(\Lambda_q - \Lambda_0) f_\zeta = f_\zeta \quad \text{on } \partial\Omega. \tag{24}$$

As we notice that the operator on the left-hand side of the boundary integral equation (24), depends on the DN map and other known quantities, we can recover the value of $w_1|_{\partial\Omega}$ by solving (24). Moreover, (24) is an inhomogeneous integral equation for f_ζ having a unique solution $f_\zeta \in H^{3/2}(\partial\Omega)$. By Fredholm alternative, the uniqueness of the solution follows from the fact that the homogeneous equation

$$-S_\zeta(\Lambda_q - \Lambda_0) f_\zeta = f_\zeta \quad \text{on } \partial\Omega,$$

only has the zero solution, which follows by its turn from the uniqueness of the CGOs.

Remark 5.2. *Nachman derived the slightly different type of boundary integral equation:*

$$e^{i\zeta.x} - (S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I)f_\zeta = f_\zeta \quad on\ \partial\Omega, \tag{25}$$

where the operator B_ζ is defined in (5). Since that we can easily show that $S_\zeta \Lambda_0 = B_\zeta + \frac{1}{2}I$, it is clear that the expressions (25) and (24) are equivalent.

Since it is complicated to check that the condition (23) is satisfied by w_1, Nachman's idea was to construct from (25) CGOs to the Schrödinger equation $(-\Delta + q)w = 0$ in $\mathbb{R}^n$, that automatically satisfy condition (23), then to prove that those CGOs coincide with the ones constructed by Sylvester and Uhlmann [31].

6. Some perspectives and open problems

In recent years, huge progress has been made by several authors in the research field of Calderón's problem, which was the motivation behind writing this paper to draw more attention to this problem to improve the known results. Due to the speed development in this topic, we note that the previous sections' results can be considered as an introduction to this domain. Therefore, a lot might lie beyond this paper. In this final section, we propose the following open questions and discuss plausibly research extensions, which can be subject to new results in several directions.

Question 6.1. (Uniqueness for $W^{1,n}$ conductivities) *For $j = 1, 2$, let $\gamma_j \in W^{1,n}(\bar{\Omega})$ be two positive functions. Then we have*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2 \quad in\ \Omega.$$

The problem of finding the lowest regularity condition on the conductivity under which uniqueness holds inspired many authors. Recently, Caro and Rogers [10] used Bourgain's spaces to prove uniqueness for Lipschitz conductivities in three and higher dimensions. Haberman [15] involves L^p harmonic analysis to show this result for $W^{1,n}$ conductivities in dimensions $n = 3, 4$, and for $W^{1+(1-\theta)(\frac{1}{2}-\frac{2}{n}),\frac{n}{1-\theta}}$, $\theta \in [0, 1)$ for $n = 5, 6$. By using the valuable tool of bilinear estimates, more improved results were given. For $\gamma \in W^{41/40+,5}$ and $\gamma \in W^{11/10+,6}$ for $n = 5$ and $n = 6$, respectively in [17]. Also, for $W^{1+\frac{n-5}{2p}+,p}$, $p \in [n, \infty)$ conductivities in five and higher dimensions [28]. The observation of those results makes us wonder how much it would be interesting to check whether it is possible to prove Brown's conjecture [7], which affirms that in three and higher dimensions $\gamma \in W^{1,n}$ is the minimum possible regularity for which uniqueness holds.

Question 6.2. (Reconstruction of Lipschitz conductivities) *If Ω is a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, $\gamma \in Lip(\Omega)$ a strictly positive real-valued function on Ω satisfying (19), with $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$, show that γ can be reconstructed on Ω from the knowledge of Λ_γ.*

We saw in the precedent section that Nachman [25] provided a constructive procedure to compute $\gamma \in C^{1,1}$ from Λ_γ . This process was followed by

García and Zhang in [13] to reconstruct C^1 or Lipschitz conductivities with $|\nabla \log \gamma|$ sufficiently small. Notice that the approaches used in [15, 17, 26] are not useful for reconstructing γ, because the proofs there are not constructive, in the meaning that they did not give a procedure to recover γ from Λ_γ.

Recently, Caro and Rogers [10] used Bourgain's spaces to prove the uniqueness for Lipschitz conductivities in three and higher dimensions. Their result makes us wonder how much it would be interesting to check whether it is possible to use this uniqueness proof to generalize Nachman's method to Lipschitz conductivities by taking off the smallness condition on $|\nabla \log \gamma|$ to improve the results in [13]. The key ingredient in the uniqueness proof in [10] for Lipschitz conductivities without a smallness condition is the following a priori estimate

$$||w||_{X_\zeta^{1/2}} \lesssim ||(-\Delta + 2\zeta.\nabla + q)w||_{X_\zeta^{-1/2}}$$

for a function $w \in \mathcal{S}(\mathbb{R}^n)$ with support in Ω, and the function spaces $X_\zeta^{\pm 1/2}$ were defined in Section 2. From the last estimate and a standard functional analysis argument, it follows a key bound on the potential q

$$||r||_{X_\zeta^{1/2}(\Omega)} \lesssim ||q||_{X_\zeta^{-1/2}},$$

for some corrector function r. The occurring complication is that the solutions here are local, but in our case, we need to extend them in some way to $\mathbb{R}^n$. Therefore, we conjecture that the techniques used until now have reached some sort of limit. Thus, we cannot follow the contraction mapping approach to apply the fixed point argument, which was used in the analysis in [25, 13]. It is rather clear, though, that this problem seems more complicated and may require new ideas beyond the known techniques to overcome its difficulties.

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OPÉRATEURS * FINIS

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Abstract. Soient H un espace de Hilbert complexe séparable de dimension infinie,

$B(H)$ l'algèbre des opérateurs linéaires bornés sur H et $\overline{W(A)}$ la fermeture de l'image numérique de l'opérateur $A \in B(H)$.

A est dit opérateur * -fini si $0 \in \overline{W(TA - AT^*)}$, $\forall\, T \in B(H)$. Cela implique qu' un opérateur A est * -fini si est seulement si: $\|TA - AT^* - \lambda I\| \geq |\lambda|$; $\forall$ $\lambda \in \mathbb{C}$et $\forall\, T \in B(H)$ .Le but de ce travail est de présenter quelques propriétés des opérateurs *-finis et donner quelques opérateurs qui appartiennent à la classe des opérateurs *-finis.

Introduction

Soient H un espace de Hilbert complexe de dimension infinie séparable et $B(H)$ l'algèbre de

tous les opérateurs linéaires bornés sur H. Soit $\overline{W(A)}$ la fermeture de l'image numérique de l'opérateur

$A \in B(H)$. A est dit opérateur * -fini si $0 \in \overline{W(TA - AT^*)}$; pour tout $T \in B(H)$.

Williams (1970) a montré que pour tout opérateur $B \in B(H)$, $0 \in \overline{W(B)}$ si et seulement, si $\|B - \lambda I\| \geq |\lambda|$; $\forall \lambda \in \mathbb{C}$. Cela implique qu'un opérateur A est * -fini si est seulement si $\|TA - AT^* - \lambda I\| \geq |\lambda|$; $\forall\, \lambda \in \mathbb{C}$et $\forall\, T \in B(H)$. La notion d'opérateur *fini a était introduite par Hamada en [..]. La définition de l'opérateur * -fini est motivée par l'étude des opérateurs finis donnés par Williams [..].

Un opérateur $A \in B(H)$ est appelé opérateur fini si est seulement si; $0 \in \overline{W(TA - AT)}$ pour tout $T \in B(H)$ ou de manière équivalente, $\|TA - AT - I\| \geq 1$ pour tout $T \in B(H)$, où I est l'opérateur identité.

Ce sujet traite la minimisation de la distance (mesurée par certaines normes) entre un commutateur variable $TT^* - T^*T$ et un opérateur fixe (Maher, 2006).

Mecheri a montré qu'un opérateur paranormal est fini. Pour plus de détails, voir (Mecheri, 2002, 2008).

Dans ce travail, nous présentons quelques propriétés des opérateurs * -finis et donner quelques opérateurs qui sont où ne sont pas des opérateurs * finis.

Préliminaire

Definition 1. *Soit* $A \in B(H)$; A *est un opérateur * -fini si* $0 \in \overline{W(TA - AT^*)}$; *pour tout* $T \in B(H)$.*(Ou* $\overline{W(TA - AT^*)}$ *est la fermeture de l'image numérique de l'opérateur* $TA - AT^*$*où;*

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$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}).$$

Qui est équivalent à:

$\|TA - AT^* - \lambda I\| \geq |\lambda|;\ \forall\ \lambda \in \mathbb{C}$et $\forall\ T \in B\,(H)$.

On note la classe des opérateurs *-finis par $F^*\,(H)$.

D'où

$$\mathcal{F}^*(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \|TA - AT^* - I\| \geq 1;\ \forall T\ \in \mathcal{B}(\mathcal{H})\}\,.$$

Remark 1. *Il est évident que l'opérateur null est *-finis.*

Remark 2. *L'opérateur identité n'est pas *-finis, alors que I est un opérateur fini. Cela implique que nous avons deux ensembles différents d'opérateurs opérateurs finis et * -opérateurs finis. La question se pose de savoir quelle est la différence entre les opérateurs finis et * -finis?*

Definition 2. *Soit $A \in B\,(H)$, le spectre reduisant approximatif de A, noté $\sigma_{ar}\,(A)$, est l'ensemble des scalairs λ tels que pour tous $\varepsilon \succ 0$, il existe un vector unitaire $x \in H : \|Ax - \lambda x\| \prec \varepsilon$ et $\left\|A^*x - \overline{\lambda}x\right\| \prec \varepsilon$.*

Definition 3. *Soit $A \in B\,(H)\,; A$ est un opérateur normaloïde si $\|A\| = r(A)$, où $r(A)$ est le rayon spectral de A, paranormal si $\|Ax\|^2 \leq \left\|A^2x\right\|\|x\|$, $\forall x \in H$ et p-hyponormal si $|A|^{2p} - |A^*|^{2p} \geq 0$ $(0 \prec p \leq 1)$.*

On a: hyponormal $\subset$ p-hyponormal $\subset$ paranormal $\subset$ normaloïde.

$A \in B\,(H)$ est log-hyponormal si A est inversible et vérifie: $log\,(A^*A) \geq \log\,(AA^*)$.

On sait que les opérateurs p-hyponormaux inversibles sont des opérateurs log-hyponormaux mais le contraire n'est pas vrai (Tanahashi, 1999). Le concepte des opérateurs log-hyponormaux est due à Ando (1972) et

Le premier article dans lequel la log-hyponormalité est apparue est Fujii, Himeji et Matsumoto (1994).

Un opérateur $A \in B\,(H)$ est de la classe (A) si: $\left|A^2\right| \blacksquare\ |A|^2$. La classe (A) a été introduite par Furuta,

Ito et Yamazaki (1998) en tant que sous-classe d'opérateurs paranormaux qui comprend les classes de

Opérateurs p-hyponormaux et log-hyponormaux. Le théorème suivant est l'un des résultats associés

avec la classe (A).

Theorem 1. *(Furuta et al., 1998)*

(1) Tout opérateur log-hyponormal est un opérateur de classe (A).
(2) Tout opérateur de classe (A) est un opérateur paranormal.

Lemma 1. *Soit $A \in B\,(H)\,;$ si A est un opérateur paranormal, alors; $\sigma_{ar}\,(A) \neq \emptyset$.*

Quelques résultats

Dans les propositions suivantes, nous présentons quelques propriétés de $F^*(H)$.

Proposition 1. *Soient $A, B \in F^*(H)$ et $\alpha \in \mathbb{C}$; alors:*

1. $\alpha A \in F^*(H)$, $\forall\ \alpha \in \mathbb{C}$.
2. $(\alpha A + I) \in F^*(H)$, $\forall\ \alpha \in \mathbb{C}$.
3. $(A + B) \in F^*(H)$.
4. $A^* \in F^*(H)$.
5. Si A est inversible; alors $A^{-1} \in F^*(H)$.

Theorem 2. *Soient $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ et $A \in \mathcal{B}(\mathcal{H})$ un opérateur de la forme:*

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

si pour tout i; $i = \overline{1,2}$: $A_{ii} \in F^*(H_i)$, alors $A \in F^*(H)$.

Theorem 3. *$F^*(H)$ est fermé sous la norme des opérateurs dans $B(H)$.*

Theorem 4. *$F^*(H)$ est invariante par equivalence unitaire*

Theorem 5. *$F^*(H)$ n'est pas dense dans $B(H)$.*

Theorem 6. *Soit $A \in B(H)$ un opérateur paranormal; alors $A - I\lambda$ est un opérateur *-fini pour tout $\lambda \in \sigma_{ra}(A)$.*

Corollary 1. *Les opérateurs suivants sous une perturbation spectrale approximative réduite sont des opérateurs* - finis:*

(1) Opérateurs hyponormaux,
(2) opérateurs p-hyponormaux,
(3) opérateurs de classe (A),
(4) opérateurs log-hyponormaux.

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LA RÉSOLUTION DU PROBLÈME DE CAUCHY ELLIPTIQUE MAL POSÉ.

MELKI HOUDEIFA

Abstract. Dans ce travail, on étudie une classe de problème mal posés en EDP, consacrée l'étude d'un problème de Cauchy elliptique mal posé.

2010 Mathematics Subject Classification. 35R30, 47A52, 35K90.

Keywords and phrases. Problèmes inverses, problèmes mal-posés, problème elliptique, problme parabolique, rgularisation, troncature spectrale, opérateurs de mollification, méthode QBV.

1. Define the problem

On note par $H = L^2((-1,1),\mathbb{R})$ l'espace de Hilbert muni du produit scalaire et la norme associe :

$$\langle u,v\rangle := \int_{-1}^{1} u(x)v(x)dx, \quad \|u\|^2 := \int_{-1}^{1} |u(x)|^2\,dx.$$

Dans le rectangle $Q = (-1,1)\times(0,T)$, on considère le problème de la chaleur non classique suivant :

$$u(x,t) - \alpha u_{xx}(x,t) - \beta u_{xx}(-x,t) = 0, x \in (-1,1), t \in (0,T), \tag{1}$$

avec les conditions aux limites périodiques :

$$u(-1,t) = u(1,t), u_x(-1,t) = u_x(1,t), t \in (0,T). \tag{2}$$

et la condition initiale :

$$u(x,0) = f(x), x \in (-1,1). \tag{3}$$

Ici α est un réel strictement positif ($\alpha > 0$) et $\beta \in \mathbb{R}$.
Ce travail est donc une continuité naturelle du travail initié par Kalmenov^{2007}[1].

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ENERGY DECAY FOR A NONLINEAR TIMOSHENKO-SYSTEM WITH INFINITE HISTORY

ZINEB KHALILI, DJAMEL OUCHENANE, AND ABDALLH EL-HAMIDI

ABSTRACT. In the present work, we consider a one-dimensional Timoshenko system with infinite history and nonlinear damping term. Where the heat conduction is given by Green and Naghdi theory. We establish the stability of the system for the case of nonequal speeds of wave propagation.

Keywords: Timoshenko system, infinite history, nonlinear damping term, energy methode.

MSC(2010): Primary: 35B40; Secondary: 74H40, 74H55, 93D20.

1. Introduction

In this work, we study the following Timoshenko system with infinite history, distributed delay and nonlinear damping term

(1.1)

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \int_0^\infty g\left(s\right)\psi_{xx}\left(x, t-s\right)ds + \alpha\left(t\right)f\left(\psi_t\right) = 0, \end{cases}$$

where $t \in (0, \infty)$ denotes the time variable and $x \in (0, 1)$ denotes the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, and ρ_1, ρ_2, b, K are positive constants, the function g is called the relaxation function. The term $\alpha(t)f(\psi_t)$ is the nonlinear damping term.

We propose the following initial and boundary conditions

(1.2)

$$\begin{cases} \varphi(x,0) = \varphi_0\left(x\right), \quad \varphi_t(x,0) = \varphi_1\left(x\right), \quad \psi(x,0) = \psi_0\left(x\right), \quad \psi_t(x,0) = \psi_1\left(x\right), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \qquad \forall t \geq 0, \end{cases}$$

where $x \in (0, 1)$.

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Iterative Continuous Collocation Method for Solving nonlinear Volterra Integro-differential Equations in the Space $S_m^{(0)}(\Pi_N)$

0.1 Introduction

In this chapter, we consider the following Volterra integro-differential equations,

$$\begin{aligned} x'(t) &= f(t) + Q(t, x(t)) + \int_0^t K(t, s, x(s), x'(s))ds, \quad t \in I = [0, T] \\ x(t_0) &= x_0 \end{aligned} \tag{1}$$

where the functions f, Q, K are sufficiently smooth. The existence and the uniqueness of the solution of (1) can be found, for example, in [4].
integro-differential equations find its applications in various fields of science and engineering. There are several numerical methods for approximating the solution of integro-differential equations are known and many different basic functions have been used. There are various methods to solve integro-differential equations such as Adomian decomposition method, successive substitutions, Laplace transformation method, Picard's method, etc (Wazwaz (2011)[59]). collocation theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly,and fast algorithms for easy implementation. collocation method have been applied for the numerical solution of different kinds of integral equations,

The aim of this chapter is to generalize the iterative continuous collocation method in [47] for construct an iterative continuous approximate solution for equation (1). In our method the approximate solution is explicit, direct and obtained by using simple iterative formulas.
In fact the applications of the iterative collocation method in the numerical analysis field possessing some of the well known advantages such as:

1. It is accurate,
2. It is possible to pick any point in the interval of integration and as well the approximate solutions and their derivatives will be applicable.
3. The method does not require discretization of the variables, and it is not affected by computation Ind off errors and one is not faced with necessity of large computer

memory and time.

The outlines of this chapter is as follows. In section 2, an iterative collocation method has been used to construct an approximate solution for (1) in the continuous spline polynomials space $S_m^{(0)}(\Pi_N)$, the convergence analysis has been given in section 3. Some numerical illustrations are provided in section 4.

0.2 Description of the collocation method

Let Π_N be a uniform partition of the interval $I = [0, T]$ with grid points $t_n = nh, \quad n = 0, ..., N-1$, where the stepsize is given by $h = \dfrac{T}{N}$. Let the collocation parameters be $0 < c_1 < ...... < c_m < 1$ and the collocation points
$\Gamma_{N,m} = \{t_{n,j} = t_n + c_j h, j = 1, ..., m, n = 0, ..., N-1\}$.
Define the subintervals $\sigma_n = [t_n, t_{n+1}]$, and $\sigma_{N-1} = [t_{N-1}, t_N]$.
Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.
We consider polynomial spline approximations $u(t)$ of the exact solution $x(t)$ in the spline space

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, ..., N-1, \}.$$

This is the space of piecewise polynomials of degree at most m. Its dimension is $Nm+1$, the same as the number of collocation points.
We seek $u \in S_m^{(0)}(\Pi_N)$ satisfies the collocation equation

$$\begin{aligned} &u'(t) = f(t) + Q(t, u(t)) + \int_0^t K(t, s, u(s), u'(s))ds, \quad t \in \Gamma_{N,m}, \\ &u(t_0) = f(0). \end{aligned} \tag{2}$$

In what follows, we consider two equivalent reformulations of problem (2) by using the function $w(t) = u'(t) \in S_{m-1}^{(-1)}(\Pi_N)$. Since $w_n \in \pi_{m-1}$, it holds for $\mu \in (0, 1]$,

$$w_n(t_n + \mu h) = \sum_{v=1}^{m} L_v(s) w_n(t_{n,v}), \tag{3}$$

$$u_n(t_n + \mu h) = f(0) + h \sum_{p=0}^{n-1} \sum_{v=1}^{m} \left(\int_0^1 L_v(\tau) d\tau \right) w_p(t_{p,v}) + h \sum_{v=1}^{m} \left(\int_0^{\mu} L_v(\tau) d\tau \right) w_n(t_{n,v}), \tag{4}$$

where $L_j(\mu) = \prod_{l \neq j}^{m} \frac{\mu - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters $c_j, j = 1, ..., m$. By using (4), the collocation equation (2) may be rewritten as the following nonlinear Volterra integro-differential equation with respect to w.

$$w_n(t) = f(t) + Q\left(t, f(0) + \int_0^t w(r)dr\right) + \int_0^t K\left(t, s, f(0) + \int_0^s w(r)dr, w(s)\right) ds$$

$$0 \leq \tau \leq s \leq t = t_{n,j}, j = 1, ..., m.$$

Hence, for each $j = 1, ..., m, n = 0, ..., N-1$, $w_n(t_{n,j})$ satisfies the following nonlinear system,

$$\begin{aligned} w_n(t_{n,j}) =& f(t_{n,j}) + Q\left(t_{n,j}, f(0) + h \sum_{p=0}^{n-1} \int_0^1 w_p(t_p + \tau h) d\tau + h \int_0^{c_j} w_n(t_n + \tau h) d\tau \right) \\ &+ h \sum_{p=0}^{n-1} \int_0^1 K\Big(t_{p,j}, t_p + \mu h, u_p(t_p + \mu h), w_p(t_p + \mu h)\Big) d\mu \\ &+ h \int_0^{c_j} K\Big(t_{n,j}, t_n + \mu h, u_n(t_n + \mu h), w_n(t_n + \mu h)\Big) d\mu, \end{aligned} \tag{5}$$

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_m^0(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1) such that

$$w_n^q(t_n + \mu h) = \sum_{v=1}^{m} L_v(\mu) w_n^q(t_{n,v}) \tag{6}$$

and

$$u_n^q(t_n + \mu h) = f(0) + h \sum_{p=0}^{n-1} \sum_{v=0}^{m} \left(\int_0^1 L_v(\tau) d\tau \right) w_p^q(t_{p,v}) + h \sum_{v=0}^{m} \left(\int_0^{\mu} L_v(\tau) d\tau \right) w_n^q(t_{n,v}), \tag{7}$$

where the coefficients $w_n^q(t_{n,j})$ are given by the following formula:

$$\begin{aligned} w_n^q(t_{n,j}) =& f(t_{n,j}) + Q\left(t_{n,j}, f(0) + h\sum_{p=0}^{n-1}\int_0^1 w_p^q(t_p+\tau h)d\tau + h\int_0^{c_j} w_n^{q-1}(t_n+\tau h)d\tau\right) \\ &+ h\sum_{p=0}^{n-1}\int_0^1 K\left(t_{p,j}, t_p+\mu h, u_p^q(t_p+\mu h), w_p^q(t_p+\mu h)\right)d\mu \\ &+ h\int_0^{c_j} K\left(t_{n,j}, t_n+\mu h, H_n^{q-1}(t_n+\mu h), w_n^{q-1}(t_n+\mu h)\right)d\mu, \end{aligned} \tag{8}$$

where,

$$H_n^q(t_n+\mu h) = f(0) + h\sum_{p=0}^{n-1}\sum_{v=0}^{m}\left(\int_0^1 L_v(\tau)d\tau\right)w_p^q(t_{p,v}) + h\sum_{v=0}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)w_n^{q-1}(t_{n,v}).$$

Such that the initial values $w_n^0(t_{n,j})$ belong in a bounded interval J.

Remark 0.2.1 *The above formula is explicit and the approximate solution u^q is given without needed to solve any algebraic system.*

In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (1), moreover, the order of convergence is m for all $q \geq m$.

0.3 Convergence analysis

In this section, we assume that the functions Q and K satisfy the following Lipschitz conditions: there exist $A_i \geq 0$ $i = 0, 1, 2$ such that

$$\begin{aligned} &|Q(t, x_1) - Q(t, x_2)| \leq A_0|x_1 - x_2|, \\ &|K(t, s, x_1, y_1) - K(t, s, x_2, y_2)| \leq A_1|x_1 - x_2| + A_2|y_1 - y_2|. \end{aligned}$$

The following lemma will be used in this section. The following result gives the existence and the uniqueness of a solution for the nonlinear system (5).

Lemma 0.3.1 *For sufficiently small h, the nonlinear system (5) has a unique solution $u \in S_m^0(I, \Pi_N)$.*

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (5) becomes

$$\begin{aligned} w_0(t_{0,j}) =& f(t_{0,j}) + Q\left(t_{0,j}, u_0 + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)w_0(t_{0,v})\right) \\ &+ h\int_0^{c_j} K\left(t_{0,j}, t_0+\mu h, u_0 + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)w_0(t_{0,v}), \sum_{v=1}^{m} L_v(\mu)w_0(t_{0,v})\right)d\mu. \end{aligned} \tag{9}$$

We consider the operator Ψ defined by:

$$\begin{aligned} \Psi : \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ x = (x_1, ..., x_m) &\longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)), \end{aligned}$$

such that for $j = 1, ..., m$, we have

$$\begin{aligned} \Psi_j(x) =& f(t_{0,j}) + Q\left(t_{0,j}, u_0 + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)x_v\right) \\ &+ h\int_0^{c_j} K\left(t_{0,j}, t_0+\mu h, u_0 + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)x_v, \sum_{v=1}^{m} L_v(\mu)x_v\right)d\mu. \end{aligned}$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\|_\infty \le hmb\,(A_0 + A_1 + A_2)\,\|x - y\|_\infty,$$

where $b = \max\{|L_v(\mu)|, \mu \in [0,1], v = 1, ..., m\}$.
Since $hmb\,(A_0 + A_1 + A_2) < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (9) has a unique solution on the interval σ_0.

(ii) Suppose that u exists and unique on the intervals $\sigma_i, i = 0, ..., n-1$ for $n \geq 1$ and we show that u exists and unique on the interval σ_n.

On the interval σ_n, the nonlinear system (5) becomes

$$w_n(t_{n,j}) = F(t_{n,j}) + Q\left(t_{n,j}, G(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)w_n(t_{n,v})\right.$$
$$\left. + h\int_0^{c_j} K\left(t_{n,j}, t_n + \mu h, R(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)w_n(t_{n,v}), \sum_{v=1}^{m} L_v(\mu)w_n(t_{n,v})\right)d\mu,\right.$$

where,

$$F(t_{n,j}) = f(t_{n,j}) + h\sum_{p=0}^{n-1}\int_0^1 K\Big(t_{p,j}, t_p + \mu h, u_p(t_p + \mu h), w_p(t_p + \mu h)\Big)d\mu.$$

$$G(t_{n,j}) = f(0) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m}\left(\int_0^1 L_v(\tau)d\tau\right)w_p(t_{p,v}).$$

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = F(t_{n,j}) + Q\left(t_{n,j}, G(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)x_v\right)$$
$$+ h\int_0^{c_j} K\left(t_{n,j}, t_n + \mu h, G(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)x_v, \sum_{v=1}^{m} L_v(\mu)x_v\right)d\mu,$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\|_\infty \leq hmb\,(A_0 + A_1 + A_2)\,\|x - y\|_\infty,$$

Since $hmb\,(A_0 + A_1 + A_2) < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (5) has a unique solution u on σ_n.

■ The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 0.3.1 *Let f, Q, K be m times continuously differentiable on their respective domains. Then for sufficiently small h, the collocation solution u converges to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$\|e^v\|_{L^\infty(I)} \leq Ch^m,$$

for $v = 0, 1$, where C is a finite constant independent of h.

Proof. Let $y = x'$. It holds that

$$y_n(t_n + \mu h) = \sum_{j=1}^{m} L_j(\mu) y_n(t_{n,j}) + \epsilon_n(\mu),\ \epsilon_n(\mu) = h^m \frac{y^m(\zeta_n(\mu))}{m!} \prod_{j=1}^{m} (\mu - c_j). \tag{10}$$

Hence,

$$\begin{aligned} x_n(t_n + \mu h) = u_0 &+ h \sum_{p=0}^{n-1} \int_0^1 \left(\sum_{v=1}^{m} L_v(\tau) y_p(t_{p,v}) + h^m \frac{y^m(\zeta_p(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j) d\tau \right) \\ &+ h \int_0^\mu \left(\sum_{v=1}^{m} L_v(\tau) y_n(t_{n,v}) + h^m \frac{y^m(\zeta_n(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j) \right) d\tau \end{aligned} \tag{11}$$

It follows that the errors $\xi = y - w$ and $e = x - u$ have the following representation

$$\xi_n(t_n + \mu h) = \sum_{j=1}^{m} L_j(\mu) \xi_n(t_{n,j}) + \epsilon_n(\mu),\ \epsilon_n(\mu) = h^m \frac{y^m(\zeta_n(\mu))}{m!} \prod_{j=1}^{m} (\mu - c_j), \tag{12}$$

$$
\begin{aligned}
e(t_n+\mu h) =& h\sum_{p=0}^{n-1}\int_0^1\left(\sum_{v=1}^{m}L_v(\tau)\xi_p(t_{p,v})+h^m\frac{y^m(\zeta_p(\tau))}{m!}\prod_{j=1}^{m}(\tau-c_j)\right)d\tau \\
&+h\int_0^{\mu}\left(\sum_{v=1}^{m}L_v(\tau)\xi_n(t_{n,v})+h^m\frac{y^m(\zeta_n(\tau))}{m!}\prod_{j=1}^{m}(\tau-c_j)\right)d\tau
\end{aligned}
\tag{13}
$$

where $\xi_n=\xi|_{\sigma_n}$ and $e_n=e|_{\sigma_n}$.

On the other hand, from (5), we have

$$
\begin{aligned}
|\xi_n(t_{n,j})| \leq& hAb\sum_{p=0}^{n}\sum_{v=1}^{m}|\xi_p(t_{p,v})| \\
&+hAb\sum_{p=0}^{n-1}\left(h\sum_{i=0}^{p-1}\sum_{v=1}^{m}|\xi_i(t_{i,v})|+h\sum_{v=1}^{m}|\xi_p(t_{p,v})|+\sum_{v=1}^{m}|\xi_p(t_{p,v})|\right) \\
&+hAb\left(h\sum_{p=0}^{n-1}\sum_{v=1}^{m}|\xi_p(t_{p,v})|+h\sum_{v=1}^{m}|\xi_n(t_{n,v})|+\sum_{v=1}^{m}|\xi_n(t_{n,v})|\right)+\alpha h^m,
\end{aligned}
\tag{14}
$$

where $A=\max\{A_i, i=0,1,2\}$ and α is a positive number.

We consider the sequence $\xi_n=\max\{|\xi_n(t_{n,v})|$ for $n=0,\dots,N-1$.

Then, from (14), ξ_n satisfies for $n=0,\dots,N-1$,

$$
\begin{aligned}
\xi_n \leq& Ahbm\sum_{p=0}^{n}\xi_p+hAbm\sum_{p=0}^{n-1}\left(h\sum_{i=0}^{p-1}\xi_i+h\xi_p+\xi_p\right) \\
&+hAbm\left(h\sum_{p=0}^{n-1}\xi_p+h\xi_n+\xi_n\right)+\alpha h^m \\
\leq& h\underbrace{Abm(2+3T)}_{\alpha_1}\sum_{p=0}^{n-1}\xi_p+h\underbrace{Abm\,(T+2)}_{\alpha_2}\xi_n+\alpha h^m.
\end{aligned}
\tag{15}
$$

Hence, for $\overline{h}<\dfrac{1}{\alpha_2}$, we have for all $h\in(0,\overline{h}]$

$$
\xi_n \leq \frac{\alpha}{1-\overline{h}\alpha_2}h^m+\frac{\alpha_1}{1-\overline{h}\alpha_2}h\sum_{p=0}^{n-1}\xi_p.
$$

Then, by Lemma **??**, for all $n = 0, ..., N-1$

$$\xi_n \leq \frac{\alpha}{1-\bar{h}\alpha_2} h^m \exp\left(\frac{T\alpha_1}{1-\bar{h}\alpha_2}\right).$$

Therefore, by using (12), we obtain

$$\begin{aligned} \|e\| &\leq mb \max\{\xi_n, n = 0, ..., N-1\} + \beta h^m \\ &\leq mb \frac{\alpha}{1-\bar{h}\alpha_2} \exp\left(\frac{T\alpha_1}{1-\bar{h}\alpha_2}\right) h^m + \beta h^m \\ &\leq \underbrace{\left(mb \frac{\alpha}{1-\bar{h}\alpha_2} \exp\left(\frac{T\alpha_1}{1-\bar{h}\alpha_2}\right) + \beta\right)}_{\alpha_3} h^m, \end{aligned}$$

where β is a positive number.

Therefore, by using (13), we obtain

$$\|e\| \leq hmb \sum_{p=0}^{n-1} \xi_p + hmb\xi_n + \gamma h^m \leq 2mbT\alpha_3 h^m + \gamma h^m,$$

where γ is a positive number,

Thus, the proof is completed by taking $C = \max(\alpha_3, 2mbT\alpha_3 + \gamma)$. ■ The following result gives the convergence of the iterative solution u^q to the exacts solution x.

Theorem 0.3.2 *Consider the iterative collocation solution u^q defined by (6), (7) and (8), then for any initial conditions $(u')^0(t_{n,j}) = w^0(t_{n,j}) \in J$ (J is a bounded interval), the iterative collocation solution u^q converges to the exact solution x. Moreover, the following error estimates hold*

$$\|(u^q)^{(v)} - x^{(v)}\| \leq Ch^m + C'\beta^q h^q$$

for $v = 0, 1$, where C, C', β are finite constants independent of h and q.

Proof. We define the error $\xi^q, e^q, \varepsilon^q$ and ς^q by $\xi^q(t) = w^q(t) - y(t)$, $e^q(t) = u^q(t) - x(t)$, $\varepsilon^q = w^q(t) - w(t)$, $\varsigma^q = u^q(t) - u(t)$ where u is defined by lemma 0.3.1.

We have, from (5) and (8), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$
\begin{aligned}
|\varepsilon_n^q(t_{n,j})| \leq & hAb\left(\sum_{p=0}^{n-1}\sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})| + \sum_{v=0}^{m}|\varepsilon_n^{q-1}(t_{n,v})|\right) \\
& + hAb\sum_{p=0}^{n-1}\left(h\sum_{i=0}^{p-1}\sum_{v=0}^{m}|\varepsilon_i^q(t_{i,v})| + h\sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})| + \sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})|\right) \\
& + hAb\left(h\sum_{p=0}^{n-1}\sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})| + h\sum_{v=0}^{m}|\varepsilon_n^{q-1}(t_{n,v})| + \sum_{v=0}^{m}|\varepsilon_n^{q-1}(t_{n,v})|\right),
\end{aligned} \tag{16}
$$

Now, for each fixed $q \geq 1$, we consider the sequence $\varepsilon_n^q = \max\{\left|\varepsilon_n^q(t_{n,v})\right| v = 1....m\}$. It follows, from (16), that for $n = 0, ..., N-1$

$$
\begin{aligned}
\varepsilon_n^q \leq hAbm\left(\sum_{p=0}^{n-1}\varepsilon_p^q + \varepsilon_n^{q-1}\right) & + hAbm\sum_{p=0}^{n-1}\left(h\sum_{i=0}^{p-1}\varepsilon_i^q + h\varepsilon_p^q + \varepsilon_p^q\right) \\
& + hAbm\left(h\sum_{p=0}^{n-1}\varepsilon_p^q + h\varepsilon_n^{q-1} + \varepsilon_n^{q-1}\right) \\
\leq h\underbrace{Abm(2+3T)}_{\alpha_1}\sum_{p=0}^{n-1}\varepsilon_p^q & + h\underbrace{Abm(2+T)}_{\alpha_2}\varepsilon_n^{q-1},
\end{aligned} \tag{17}
$$

We consider the sequence $\eta^q = \max\{\varepsilon_n^q, n = 0, ..., N-1\}$ for $q \geq 1$.
Then, from (17), we obtain

$$
\varepsilon_n^q \leq \alpha_1 h\sum_{p=0}^{n-1}\varepsilon_p^q + \alpha_2 h\eta^{q-1}. \tag{18}
$$

Hence, by Lemma **??**, for all $n = 0, ..., N-1$

$$
\eta^q \leq \underbrace{\alpha_2\exp(\alpha_1 T)}_{\beta} h\eta^{q-1} \leq \beta^2 h^2\eta^{q-2} \leq ... \leq \beta^q h^q\eta^0.
$$

Since, $w^0(t_{n,j}) \in J$ (bounded interval) and w is bounded by Lemma 0.3.1, then there exists $\delta > 0$ such that $\eta^0 < \delta$, which implies that, for all $q \geq 1$

$$\eta^q \leq \delta\beta^q h^q.$$

Therefore, by using (3) and (6), we obtain

$$\|\varepsilon^q\| \leq mb\eta^q \leq \underbrace{mb\delta}_{d}\beta^q h^q,$$

Hence, by Theorem (0.3.1), we deduce that

$$\|\xi^q\| \leq \|\varepsilon^q\| + \|w - y\| \leq d\beta^q h^q + Ch^m.$$

On the other hand, from (4) and (7), we have

$$\|\varsigma^q\| \leq 2Tmb\|\varepsilon^q\| \leq 2Tmbd\beta^q h^q.$$

Finally, by using Theorem (0.3.1), we deduce that

$$\|e^q\| \leq \|\varsigma^q\| + \|u - x\| \leq \underbrace{2Tmbd}_{d'}\beta^q h^q + Ch^m.$$

Thus, the proof is completed by taking $C' = \max(d, d')$. ■

0.4 Numerical Examples

In order to test the applicability of the presented method, we consider the following examples with $T = 1$. These examples have been solved with various values of N, m and $q = m$. We used the collocation parameters $c_i = \frac{i}{m+1}, i = 1, ..., m$. In each example, we calculate the error between x and the iterative collocation solution u^m.

The absolute errors at the particular points are given to compare our solutions with the solutions obtained by [53, 56].

The results in these examples confirm the theoretical results; moreover, the results obtained by the present method is very superior to that obtained by the methods in [53, 56].

Example 0.4.1 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t \cos(t + s + x(s) + x'(s)) + \frac{1}{1 + x^2(s)} ds, \ t \in [0, 1].$$

with f *is chosen so that the exact solution is* $x(t) = 2t + 5$*. The absolute errors for* $(N, m) \in \{(2, 3), (4, 3), (4, 4), (6, 4)\}$ *at* $t = 0, 0.1, ..., 1$ *are presented in Table 0.1. From the Table 0.1, we note that the absolute error reduces as N or m increases.*

Table 0.1: Absolute errors for Example 0.4.1

t	$N = 2$ $m = 3$	$N = 4$ $m = 3$	$N = 4$ $m = 4$	$N = 6$ $m = 4$
0	0.0	0.0	0.0	0.0
0.1	2.03 E −4	1.48 E −5	4.40 E −7	4.99 E −8
0.2	2.37 E −4	1.73 E −5	1.36 E −6	2.90 E −7
0.3	2.23 E −4	4.94 E −7	1.43 E −6	2.27 E −7
0.4	2.88 E −4	7.86 E −6	1.21 E −6	9.56 E −7
0.5	5.53 E −4	6.69 E −5	8.55 E −6	9.58 E −7
0.6	1.07 E −3	6.17 E −5	7.95 E −6	8.30 E −7
0.7	1.12 E −3	5.43 E −5	7.46 E −6	7.60 E −7
0.8	1.14 E −3	6.04 E −5	7.24 E −6	6.97 E −7
0.9	1.52 E −3	6.90 E −5	6.36 E −6	3.84 E −7
1	2.71 E −3	1.38 E −4	6.64 E −7	4.33 E −7

Example 0.4.2 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t \frac{\cos(t)}{1+t+(x'(s))^2} + \frac{t\sin(s)}{2+x^2(s)} ds, \ t \in [0,1].$$

with f is chosen so that the exact solution is $x(t) = 3\cos(t) + 1$. The absolute errors for $(N, m) \in \{(2,3),(4,3),(4,4),(6,4)\}$ at $t = 0, 0.1, ..., 1$ are presented in Table 0.2. From the Table 0.2, we note that the absolute error reduces as N or m increases.

Table 0.2: Absolute errors for Example 0.4.2

t	$N=2$ $m=3$	$N=4$ $m=3$	$N=4$ $m=4$	$N=6$ $m=4$
0	0.0	0.0	0.0	0.0
0.1	6.87 E −4	5.08 E −5	1.33 E −7	1.13 E −8
0.2	8.11 E −4	5.17 E −5	1.36 E −7	4.96 E −8
0.3	8.13 E −4	8.67 E −5	5.64 E −7	6.11 E −8
0.4	8.41 E −4	9.49 E −5	6.16 E −7	1.38 E −7
0.5	7.58 E −4	9.41 E −5	7.46 E −7	1.61 E −7
0.6	1.28 E −3	1.37 E −4	1.43 E −6	2.44 E −7
0.7	1.38 E −3	1.40 E −4	1.46 E −6	3.58 E −7
0.8	1.39 E −3	1.65 E −4	2.38 E −6	3.69 E −7
0.9	1.41 E −3	1.71 E −4	2.47 E −6	5.06 E −7
1	1.35 E −3	1.69 E −4	2.61 E −6	5.13 E −7

Example 0.4.3 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t (ts\arctan(s + x(s) + x'(s)) + \cos(t - s + x(s)))ds, \; t \in [0, 1].$$

with f is chosen so that the exact solution is $x(t) = 2t + 1$. The absolute errors for $(N, m) \in \{(2, 2), (2, 3), (4, 3), (4, 4)\}$ at $t = 0, 0.1, ..., 1$ are presented in Table 0.2. From the Table 0.3, we note that the absolute error reduces as N or m increases.

Table 0.3: Absolute errors for Example 0.4.3

t	$N = 2$, $m = 2$	$N = 2$, $m = 3$	$N = 4$, $m = 3$	$N = 4$, $m = 4$
0	0.0	0.0	0.0	0.0
0.1	1.26 E −4	5.86 E −7	1.40 E −8	3.03 E −9
0.2	2.47 E −4	1.58 E −6	5.66 E −8	8.11 E −9
0.3	3.61 E −4	3.42 E −6	1.00 E −7	1.49 E −9
0.4	4.70 E −4	6.53 E −6	1.13 E −7	6.14 E −9
0.5	5.73 E −4	1.13 E −5	1.47 E −7	2.00 E −9
0.6	5.66 E −4	1.13 E −5	1.43 E −7	2.16 E −9
0.7	5.63 E −4	1.11 E −5	1.41 E −7	1.01 E −8
0.8	5.66 E −4	1.06 E −5	1.31 E −7	4.96 E −10
0.9	5.73 E −4	1.01 E −5	1.26 E −7	1.76 E −10
1	5.84 E −4	9.46 E −6	1.22 E −7	2.00 E −9

Example 0.4.4 *([53, 56]) Consider the linear Volterra integro-differential equation given by*

$$x'(t) = 1 - \int_0^t x(s)ds, \; t \in [0, 1].$$

with the initial conditions $x(0) = 0$ *and the exact solution* $x(t) = \sin(t)$. *Here,* $f(t) = 1, g(t) = 0, K(t, s) = -1$.

The absolute errors for $N = 6, 10$ *and* $m = q = 5$ *at* $t = 0, 0.1, ..., 1$ *are displayed in Table 0.4. The numerical results of the absolute error function obtained by the present method are compared in Table 0.4 with the absolute error function of the Taylor method given in [53] and Bessel method [56] for an approximate polynomial solutions of degree* 5.

Table 0.4: Comparison of the absolute errors of Example 4. 4.4

t	Taylor method [53]	Bessel method [56]	Present method $N = 6$	Present method $N = 10$
0.0	0.0	0.0	0.0	0.0
0.1	2.00 E −11	2.49 E −7	1.58 E −9	1.26 E −10
0.2	2.50 E −9	4.02 E −7	5.45 E −10	9.50 E −11
0.3	4.33 E −8	3.00 E −7	2.71 E −9	2.61 E −10
0.4	3.24 E −7	2.05 E −7	8.90 E −10	1.08 E −10
0.5	1.54 E −6	2.83 E −7	5.60 E −9	2.04 E −10
0.6	5.52 E −6	3.75 E −7	7.20 E −9	4.96 E −12
0.7	1.62 E −5	1.65 E −7	1.19 E −9	4.62 E −10
0.8	4.12 E −5	1.81 E −7	1.36 E −10	5.00 E −10
0.9	9.38 E −5	1.18 E −6	1.27 E −9	1.72 E −10
1.0	1.95 E −4	9.66 E −6	7.85 E −9	9.92 E −10

0.5 Conclusion

In this chapter, we proposed the iterative collocation method for the numerical solution of integro-differential equations(1) in the spline space $S_m^{(0)}(\Pi_N)$. Our numerical results are compared with exact solutions and existing methods. Error analysis shows the accuracy and effectiveness of the proposed scheme Hence, the present method is approached through the illustrative examples which show the efficiency, validity and applicability.

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EXISTENCE AND UNIQUENESS RESULTS FOR NONLOCAL BOUNDARY VALUE PROBLEM FOR IMPLICIT FRACTIONAL q-DIFFERENCE EQUATIONS

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Abstract. In this paper, we study the existence and uniqueness of solutions for nonlinear implicit Caputo fractional q-difference equations with nonlocal conditions. We convert the given fractional q -difference equations into an equivalent integral equation. Then we construct appropriate mappings and employ the Krasnoselskii's fixed point theorem to show the existence of a solution. We also use the Banach fixed point theorem to show the existence of a unique solution. Finally, an example is given to illustrate our results.

Key words and phrases. Fractional q-difference equations; Caputo fractional q-difference derivatives; Fixed point theorems; Existence; Uniqueness.

1. Introduction

Fractional differential equations without and with delay arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of nonlinear and linear fractional differential equations without and with delay have received the attention of many authors, see [5]-[12], [16], [20]-[23], [27], [30] and the references therein.

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [17, 18]. Basic definitions and properties of q-difference calculus can be found in the book mentioned in [19]. The problems of nonlinear fractional q-difference equations have aroused considerable attention. Many people pay attention to the existence and multiplicity of solutions or positive solutions for problems of nonlinear fractional q-difference equations, see [1]-[4], [13]-[15], [17], [18], [27], [24], [28], [29].

In [15], Ferreira investigate the existence of nontrivial solutions to the nonlinear q-fractional boundary value problem

$$\begin{cases} D_q^\alpha u(t) = -f(t, u(t)), \ 0 \leq t \leq 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $0 < q < 1$, D_q^α denotes the Riemann-Liouville fractional q-derivative of order $1 < \alpha \leq 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given nonnegative continuous function.

In [28], Yang study the existence and uniqueness results for anti-periodic boundary value problems involving nonlinear fractional q-difference equations given by

$$\begin{cases} {}^C D_q^\alpha u(t) = f(t, u(t)), \ 0 \leq t \leq 1, \\ u(0) = -u(1), \ D_q u(0) = -D_q u(1), \end{cases}$$

where $0 < q < 1$, ${}^C D_q^\alpha$ denotes the Caputo fractional q-derivative of order $1 < \alpha \leq 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In [1], the authors investigated the existence, uniqueness and Ulam stability of solutions for the following implicit fractional q-difference equation

$$\begin{cases} {}^C D_q^\alpha u(t) = f\left(t, u(t), {}^C D_q^\alpha u(t)\right), \ 0 \leq t \leq T, \\ u(0) = u_0 \in \mathbb{R}, \end{cases}$$

where $0 < q < 1$, $0 < \alpha \leq 1$, $T > 0$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Inspired and motivated by the above works, we study the existence and uniqueness of solutions for the following implicit fractional q-difference equation with nonlocal conditions

$$\begin{cases} {}^C D_q^\alpha u(t) = f\left(t, u(t), {}^C D_q^\alpha u(t)\right), \ 0 \leq t \leq T, \\ u(0) + g(u) = u_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $0 < q < 1$, $0 < \alpha \leq 1$, $T > 0$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions. In passing, we note that the application of nonlocal condition $u(0) + g(u) = u_0$ in physical problems yields better effect than the initial condition $u(0) = u_0$, see [11]. To show the existence and uniqueness of solutions, we transform (1) into an integral equation and then use the Krasnoselskii and Banach fixed point theorems.

The rest of this paper is organized as follows. In Section 2 we introduce some notations and lemmas, and state some preliminaries results needed in later sections. In Section 3, we prove the existence and uniqueness of solutions for the problem (1). Finally, an example is given in Section 4 to illustrate our results.

2. Preliminaries

In this section, we recall some basic definitions and necessary lemmas related to fractional q-calculus and fixed point theorems that will be used throughout this paper.

Let $J=[0,T]$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\| = \sup_{t\in J} |u(t)|.$$

As usual $L^1(J)$ denotes the space of measurable functions $u: J\to\mathbb{R}$ which are Lebesgue integrable with the norm

$$\|u\|_1 = \int_J |u(t)|\,dt.$$

Let us recall some definitions and properties of fractional q-calculus. For $a\in\mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q analogue of the power $(a-b)^n$ is

$$(a-b)^{(0)} = 1,\ \ (a-b)^{(n)} = \prod_{k=0}^{n-1}\left(a-bq^k\right),\ a,b\in\mathbb{R},\ n\in\mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right),\ a,b,\alpha\in\mathbb{R}.$$

Definition 1 ([19])**.** *The q-gamma function is defined by*

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}},\ \xi\in\mathbb{R}\backslash\{0,-1,-2,...\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q\,\Gamma_q(\xi)$.

Definition 2 ([19])**.** *The q-derivative of order $n\in\mathbb{N}$ of a function $u: J\to\mathbb{R}$ is defined by $D_q^0 u(t) = u(t)$,*

$$D_q u(t) = D_q^1 u(t) = \frac{u(t)-u(qt)}{(1-q)\,t},\ t\neq 0,\ D_q u(0) = \lim_{t\to 0} D_q u(t),$$

and

$$D_q^n u(t) = D_q D_q^{n-1} u(t),\ t\in J,\ n\in\{1,2,...\}.$$

Set $J_t = \{tq^n : n\in\mathbb{N}\}\cup\{0\}$.

Definition 3 ([19])**.** *The q-integral of a function $u : J_t \to \mathbb{R}$ is defined by*

$$I_q u(t) = \int_0^t u(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f(tq^n),$$

provided that the series converges.

We note that $D_q I_q u(t) = u(t)$, while u is continuous at 0, then

$$I_q D_q u(t) = u(t) - u(0).$$

Definition 4 ([2])**.** *The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}^+$ of a function $u : J \to \mathbb{R}$ is defined by $I_q^0 u(t) = u(t)$ and*

$$I_q^{\alpha} u(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s)\, d_q s,\ t \in J.$$

Lemma 1 ([24])**.** *For $\alpha \in \mathbb{R}^+$ and $\lambda \in (1, \infty)$ we have*

$$I_q^{\alpha} (t-a)^{(\lambda)} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda+\alpha)} (t-a)^{(\lambda+\alpha)},\ 0 < a < t.$$

In particular

$$I_q^{\alpha} 1 = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 5 ([25])**.** *The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}^+$ of a function $u : J \to \mathbb{R}$ is defined by $D_q^0 u(t) = u(t)$ and*

$$D_q^{\alpha} u(t) = D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u(t),\ t \in J,$$

where $[\alpha]$ is the integer part of α.

Definition 6 ([25])**.** *The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}^+$ of a function $u : J \to \mathbb{R}$ is defined by ${}^C D_q^0 u(t) = u(t)$, and*

$${}^C D_q^{\alpha} u(t) = I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u(t),\ t \in J.$$

Lemma 2 ([25])**.** *Let $\alpha \in \mathbb{R}^+$. Then the following equality holds*

$$I_q^{\alpha C} D_q^{\alpha} u(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} D_q^{\alpha} u(0).$$

In particular, if $\alpha \in (0,1)$, then

$$I_q^{\alpha C} D_q^{\alpha} u(t) = u(t) - u(0).$$

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a solution of (1).

Theorem 1 (Banach's fixed point theorem [26])**.** *Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2 (Krasnoselskii's fixed point theorem [26])**.** *Let Ω be a non-empty bounded closed convex subset of a Banach space $(S, \|.\|)$. Suppose that F_1 and F_2 map Ω into S such that*
(i) $F_1u + F_2v \in \Omega$ for all $u, v \in \Omega$,
(ii) F_1 is continuous and compact,
(iii) F_2 is a contraction.
Then there is a $u \in \Omega$ with $F_1u + F_2u = u$.

3. Existence and uniqueness

Let us start by defining what we mean by a solution of the problem (1).

Definition 7. *A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of problem (1) if u satisfies ${}^C D_q^\alpha u(t) = f\left(t, u(t), {}^C D_q^\alpha u(t)\right)$ for any $t \in J$ and $u(0) + g(u) = u_0$.*

For the existence of solutions for the problem (1), we need the following auxiliary lemma.

Lemma 3. *Let $u \in C(J, \mathbb{R})$ and u' exists, then u is a solution of the initial value problem (1) if and only if it is a solution of the integral equation*

$$u(t) = u_0 - g(u) + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, u(s), {}^C D_q^\alpha u(s)\right) d_q s. \tag{2}$$

Proof. Suppose u satisfies the problem (1). First we write (1) as

$$I_q^{\alpha C} D_q^\alpha u(t) = I_q^\alpha f\left(t, x(t), {}^C D_{0^+}^\alpha x(t)\right).$$

In view of Lemma 2, we have

$$u(t) - u(0) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, u(s), {}^C D_q^\alpha u(s)\right) d_q s.$$

The condition $u(0) + g(u) = u_0$ implies that

$$u(t) = u_0 - g(u) + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, u(s), {}^C D_q^\alpha u(s)\right) d_q s.$$

Then, we obtain (2). Since each step is reversible, the converse follows easily. This completes the proof. □

In the following subsections we prove existence, as well as existence and uniqueness results for the problem (1) by using a variety of fixed point theorems.

The following assumptions will be used in our main results

(H1) There exist constants $L_1 \in \mathbb{R}^+$ and $L_2 \in (0,1)$ such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|,$$

for $t \in J$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(H2) There exists a constant $L_g \in (0,1)$ such that

$$|g(u) - g(v)| \leq L_g \|u - v\|,$$

for $t \in J$ and $u, v \in C(J, \mathbb{R})$.

3.1. **Existence and uniqueness results via Banach's fixed point theorem.**

Theorem 3. *Assume that the assumptions (H1)-(H2) are satisfied. If*

$$L_g + \frac{L_1 T^\alpha}{(1 - L_2)\Gamma_q(\alpha + 1)} < 1. \tag{3}$$

Then there exists a unique solution for the problem (1) on J.

Proof. We define the operator $\Phi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$(\Phi u)(t) = u_0 - g(u) + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h_u(s)\, d_q s,$$

where $h_u \in C(J, \mathbb{R})$ be such that

$$h_u(t) = f(t, u(t), h_u(t)).$$

Clearly, the fixed points of operator Φ are solutions of problem (1).

For any $u, v \in C(J, \mathbb{R})$ and $t \in J$, we have

$$\begin{aligned} &|(\Phi u)(t) - (\Phi v)(t)| \\ &\leq |g(u) - g(v)| + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |h_u(s) - h_v(s)|\, d_q s \\ &\leq L_g \|u - v\| + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |h_u(s) - h_v(s)|\, d_q s, \end{aligned} \tag{4}$$

where $h_v \in C(J, \mathbb{R})$ be such that

$$h_v(t) = f(t, v(t), h_v(t)).$$

By (H1), we have

$$\begin{aligned}\left|h_u(t)-h_v(t)\right| &= \left|f\left(t,u(t),h_u(t)\right)-f\left(t,v(t),h_v(t)\right)\right| \\ &\le L_1\left|u(t)-v(t)\right|+L_2\left|h_u(t)-h_v(t)\right| \\ &\le \frac{L_1}{1-L_2}\left|u(t)-v(t)\right|. \qquad (5)\end{aligned}$$

By replacing (5) in the inequality (4), we get

$$\begin{aligned}&\left|(\Phi u)(t)-(\Phi v)(t)\right| \\ &\le L_g\left\|u-v\right\|+\frac{L_1}{1-L_2}\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|u(t)-v(t)\right|d_qs \\ &\le\left(L_g+\frac{L_1T^\alpha}{(1-L_2)\Gamma_q(\alpha+1)}\right)\left\|u-v\right\|.\end{aligned}$$

Therefore

$$\left\|\Phi u-\Phi v\right\|\le\left(L_g+\frac{L_1T^\alpha}{(1-L_2)\Gamma_q(\alpha+1)}\right)\left\|u-v\right\|.$$

From (3), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem (1) on J. □

3.2. **Existence results via Krasnoselskii's fixed point theorem.**

Theorem 4. *Assume (H2) and the following hypothesis*

(H3) There exist functions $p_i\in C(J,\mathbb{R}^+)$, $i=1,...,3$ with $p_3^*=\sup_{t\in J}p_3(t)<1$ such that

$$\left|f(t,u,v)\right|\le p_1(t)+p_2(t)\left|u\right|+p_3(t)\left|v\right|,\ \textit{for each } t\in J \textit{ and } u,v\in\mathbb{R}.$$

If

$$\frac{p_2^*T^\alpha}{(1-L_g)(1-p_3^*)\Gamma_q(\alpha+1)}<1,$$

where $p_2^*=\sup_{t\in J}p_2(t)$. Then the boundary value problem (1) has at least one solution.

Proof. Let us fix

$$\rho\ge R\Lambda,$$

where

$$R=\frac{(1-p_3^*)\Gamma_q(\alpha+1)}{(1-L_g)(1-p_3^*)\Gamma_q(\alpha+1)-p_2^*T^\alpha},$$

and

$$\Lambda=\left|u_0\right|+\left|g(0)\right|+\frac{p_1^*}{1-p_3^*}\frac{T^\alpha}{\Gamma_q(\alpha+1)},$$

with $p_1^* = \sup_{t\in J} p_1(t)$. Consider the non-empty bounded closed convex subset $\Omega = \{u \in C(J,\mathbb{R}),\ \|u\| \le \rho\}$, and define two operators F_1 and F_2 on Ω as follows

$$(F_1u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h_u(s)\, d_qs,$$

and

$$(F_2u)(t) = u_0 - g(u).$$

We shall use the Krasnoselskii fixed point theorem to prove there exists at least one fixed point of the operator $F_1 + F_2$ in Ω. The proof will be given in several steps.

Step 1. We prove that $F_1u + F_2v \in \Omega$ for all $u, v \in \Omega$.

For any $u, v \in \Omega$, we have

$$\begin{aligned}&|(F_1u)(t) + (F_2v)(t)|\\ &\le |u_0| + |g(v)| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |h_u(s)|\, d_qs. \end{aligned} \tag{6}$$

By (H2) and (H3), we have

$$\begin{aligned}|g(v)| &\le |g(v) - g(0)| + |g(0)|\\ &\le L_g\|v\| + |g(0)|\\ &\le L_g\rho + |g(0)|,\end{aligned} \tag{7}$$

and

$$\begin{aligned}|h_u(t)| &= |f(t, u(t), h_u(t))|\\ &\le p_1(t) + p_2(t)|u(t)| + p_3(t)|h_u(t)|\\ &\le \frac{p_1^*}{1-p_3^*} + \frac{p_2^*}{1-p_3^*}|u(t)|.\end{aligned} \tag{8}$$

By replacing (7) and (8) in the inequality (6), we get

$$\begin{aligned}&|(F_1u)(t) + (F_2v)(t)|\\ &\le |u_0| + L_g\rho + |g(0)| + \left(\frac{p_1^*}{1-p_3^*} + \frac{p_2^*}{1-p_3^*}\rho\right)\frac{T^\alpha}{\Gamma_q(\alpha+1)}.\end{aligned}$$

Thus

$$\begin{aligned}\|F_1u + F_2v\| &\le \Lambda + \left(L_g + \frac{p_2^*}{1-p_3^*}\frac{T^\alpha}{\Gamma_q(\alpha+1)}\right)\rho\\ &\le \frac{\rho}{R} + \left(1 - \frac{1}{R}\right)\rho = \rho.\end{aligned}$$

Hence, $F_1u + F_2v \in \Omega$, for all $u, v \in \Omega$.

Step 2. We prove that F_1 is compact.

For all $u \in \Omega$ and by (8), we have

$$\begin{aligned}\left|\left(F_1 u\right)(t)\right| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|h_u(s)\right| d_q s \\ &\leq \left(\frac{p_1^*}{1-p_3^*}+\frac{p_2^*}{1-p_3^*}\rho\right)\frac{T^\alpha}{\Gamma_q(\alpha+1)},\end{aligned}$$

thus

$$\left\|F_1 u\right\| \leq \left(\frac{p_1^*}{1-p_3^*}+\frac{p_2^*}{1-p_3^*}\rho\right)\frac{T^\alpha}{\Gamma_q(\alpha+1)}.$$

Hence, F_1 is uniformly bounded on Ω.

It remains to show that $F_1(\Omega)$ is equicontinuous, let $x \in \Omega$, then for any $0 < t_1 < t_2 \leq T$ and from (8), we have

$$\begin{aligned}&\left|\left(F_1 u\right)(t_2)-\left(F_1 u\right)(t_1)\right| \\ &= \left|\int_0^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h_u(s)\,d_q s\tau - \int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}h_u(s)\,d_q s\right| \\ &\leq \frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|(t_2-qs)^{\alpha-1}-(t_1-qs)^{\alpha-1}\right|\left|h_u(s)\right| d_q s \\ &+ \frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}\left|h_u(s)\right| d_q s \\ &\leq \frac{1}{\Gamma_q(\alpha)}\left(\frac{p_1^*}{1-p_3^*}+\frac{p_2^*}{1-p_3^*}\rho\right)\int_0^{t_1}(t_1-qs)^{\alpha-1}-(t_2-qs)^{\alpha-1}\,d_q s \\ &+ \frac{1}{\Gamma_q(\alpha)}\left(\frac{p_1^*}{1-p_3^*}+\frac{p_2^*}{1-p_3^*}\rho\right)\int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}\,d_q s \\ &\leq \frac{1}{\Gamma_q(\alpha+1)}\left(\frac{p_1^*}{1-p_3^*}+\frac{p_2^*}{1-p_3^*}\rho\right)\left(2(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\right)\end{aligned}$$

(9)

$$\leq \frac{2}{\Gamma_q(\alpha+1)}\left(\frac{p_1^*}{1-p_3^*}+\frac{p_2^*}{1-p_3^*}\rho\right)(t_2-t_1)^\alpha.$$

As $t_1 \to t_2$, the right-hand side of inequality (9) tends to zero and the convergence is independent of u in Ω, which means that $F_1(\Omega)$ is equicontinuous. The Arzela-Ascoli theorem implies that F_1 is compact.

Step 3. We show that F_1 is continuous.

Let (u_n) be a sequence such that $u_n \to u$ in Ω, we have

$$\left|\left(F_1 u_n\right)(t)-\left(F_1 u\right)(t)\right| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|h_{u_n}(s)-h_u(s)\right| d_q s, \tag{10}$$

where $h_{u_n} \in C(J,\mathbb{R})$ be such that

$$h_{u_n}(t) = f(t, u_n(t), h_{u_n}(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous function, we get

$$h_{u_n}(t) \to h_u(t) \text{ as } n \to \infty.$$

Then, the Lebesgue dominated convergence theorem implies that

$$|(F_1 u_n)(t) - (F_1 u)(t)| \to 0 \text{ as } n \to \infty.$$

This shows that $(F_1 u_n)$ converges pointwise to $F_1 u$ on J. Moreover, the sequence $(F_1 u_n)$ is equicontinuous by a similar proof of Step 2. Therefore $(F_1 u_n)$ converges uniformly to $F_1 u$ and hence F_1 is continuous.

Step 4. We prove that $F_2 : \Omega \to C(J,\mathbb{R})$ is a contraction mapping. For all $u, v \in \Omega$ and $t \in J$, we have

$$\begin{aligned} |(F_2 u)(t) - (F_2 v)(t)| &= |g(u) - g(v)| \\ &\leq L_g \|u - v\|, \end{aligned}$$

thus

$$\|F_2 u - F_2 v\| \leq L_g \|u - v\|.$$

Hence, the operator F_2 is a contraction.

Clearly, all the hypotheses of the Krasnoselskii fixed point theorem are satisfied. Hence, there a fixed point $u \in \Omega$ such that $u = F_1 u + F_2 u$ which is a solution of the problem (1). □

Example 1. *We consider the following problem of implicit fractional* $\frac{1}{2}$*-difference equations*

$$\text{(11)} \qquad \begin{cases} D_{\frac{1}{2}}^{\frac{1}{2}} u(t) = \frac{1}{(\exp(t)+3)\left(1+|u(t)|+\left|{}^C D_{\frac{1}{2}}^{\frac{1}{2}} u(t)\right|\right)}, \ t \in J = [0,1], \\ u(0) - \sum_{i=1}^{n} c_i u(t_i) = 1, \end{cases}$$

where $T = 1$, $\alpha = q = \frac{1}{2}$, $u_0 = 1$, $0 < t_1 < ... < t_n < 1$ *and* c_i, $i = 1, ..., n$ *are positive constants with*

$$\sum_{i=1}^{n} c_i \leq \frac{1}{6}.$$

Set

$$\begin{aligned} f(t, u_1, v_1) &= \frac{1}{(\exp(t)+3)(1+|u_1|+|v_1|)}, \ t \in J, \ u_1, v_1 \in \mathbb{R}, \\ g(u) &= -\sum_{i=1}^{n} c_i u(t_i), \ u \in C(J,\mathbb{R}). \end{aligned}$$

For each $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $t \in J$, we have

$$\begin{aligned}
&\left|f\left(t, u_1, v_1\right)-f\left(t, u_2, v_2\right)\right| \\
&=\left|\frac{1}{(\exp (t)+3)}\left(\frac{1}{\left(1+\left|u_1\right|+\left|v_1\right|\right)}-\frac{1}{\left(1+\left|u_2\right|+\left|v_2\right|\right)}\right)\right| \\
&\leq \frac{\left|u_1-u_2\right|+\left|v_1-v_2\right|}{(\exp (t)+3)\left(1+\left|u_1\right|+\left|v_1\right|\right)\left(1+\left|u_2\right|+\left|v_2\right|\right)} \\
&\leq \frac{1}{4}\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right) .
\end{aligned}$$

And for each $u, v \in C(J, \mathbb{R})$, we get

$$\begin{aligned}
|g(u)-g(v)| & \leq \sum_{i=1}^n c_i\left|u\left(t_i\right)-v\left(t_i\right)\right| \\
& \leq \sum_{i=1}^n c_i\|u-v\| \leq \frac{1}{6}\|u-v\| .
\end{aligned}$$

Hence, conditions (H1) and (H2) are satisfied with $L_1 = L_2 = \frac{1}{4}$ and $L_g = \frac{1}{6}$. The condition

$$L_g+\frac{L_1 T^\alpha}{\left(1-L_2\right) \Gamma_q(\alpha+1)}=\frac{1}{6}+\frac{\frac{1}{4}}{\left(1-\frac{1}{4}\right) \Gamma_q\left(\frac{3}{2}\right)} \simeq 0.49<1,$$

is satisfied. It follows from Theorem 3 that the problem (1) has a unique solution on J.

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A VARIATIONAL STUDY FOR A CONTACT PROBLEM BETWEEN TWO THERMO-VISCOELASTIC BODIES

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Abstract

We study a mathematical model wich describes the bilateral contact problem with wear and damage between two thermo-viscoelastic bodies. The contact is frictional and bilateral .Wich results in the wear and damage of contacting surface. The evolution of the wear function is described with Archard's law.The evolution of the damage is described by an inclusion of parabolic type. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic iné qualities, differentiel equations and fixed point argument.

1 Introduction

The modelisation of a contact phenomenon is determined by a set of assumptions influencing on the form and structure of partial differential equations system or on boundary conditions of the associated mathematical model.

Among the assumptions influencing the partial differential equations system :

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-*Hypothesis about the geometry of the deformation* (small deformation or others).

-*Hypothesis about the mechanical process* (quasi-static or dynamic).

-*Hypothesis about the laws of material behavior* (elastic, viscoelastic,...).

The model equations can be influenced by additional phenomena (thermal, piezoelectric,...).

The boundary conditions on the contact surface are described in both normal direction and in the tangential plane, these are called boundary conditions of friction.

In the direction of normal, we have unilateral and bilateral contact (when there is no separation between the body and the obstacle). The normal compliance (when the obstacle is deformable).

The boundary conditions are also influenced by the consideration of various underlying phenomena accompanying the contact with friction, adhesion, wear, thermal effects, the dependence of the threshold friction versus slip or the slip speed can influence the boundary conditions of the mathematical model.

Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes are just a few examples. The constitutive laws with internal variables has been used in various publications in order to model the effect of internal variables in the behavior of real bodies. Some of the internal state variables considered by many authors are the temperature and the damage field.

Wear is one of the processes which reduce the lifetime of modern machine elements. It represents the unwanted removal of materials from surfaces of contacting bodies occurring in relative motion.

In this paper we consider a mathematical frictionel contact, between two thermo-viscoelastic bodies, with damage and wear. For this, we consider rate-type constitutive equation for bodies of the form

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell), \zeta^\ell\big) + \mathcal{F}^\ell(\theta^\ell, \zeta^\ell), \quad \text{in } \Omega^\ell \times (0,T) \tag{1.1}$$

In which $\mathbf{u}^\ell$, $\boldsymbol{\sigma}^\ell$represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ^ℓ represents the temperature, ζ^ℓ is the damage field, $\mathcal{A}^\ell$ and $\mathcal{G}^\ell$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{F}^\ell$ are nonlinear constitutive functions which describe the behavior of the material. The differential inclusion used for the evolution of the damage field is

$$\dot{\zeta}^\ell - \kappa^\ell \Delta\zeta^\ell + \partial\psi_{K^\ell}(\zeta^\ell) \ni \phi^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell), \zeta^\ell\big) \quad \text{in } \Omega^\ell \times (0,T), \tag{1.2}$$

Where $\psi_{K^\ell}(\zeta^\ell)$ denotes the subdifferential of the indicator function of the set K^ℓ of admissible damage functions defined by

$$K^\ell = \{\zeta \in H^1(\Omega^\ell); 0 \le \zeta \le 1, \text{ a.e. in } \Omega^\ell\}, \tag{1.3}$$

and ϕ^ℓ are given constitutive functions which describe the sources of the damage in the system. When $\zeta^\ell = 0$ the material is completely damaged, when $\zeta^\ell = 1$ the material is undamaged, , and for $0 < \zeta^\ell < 1$ there is partial damage.

The thermo-viscoelastic constitutive law (1.1) includes a temperature effects described by the parabolic aquation given by

$$\dot{\theta}^\ell - \kappa_0 \Delta\theta^\ell = \psi^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\mathbf{u}^\ell), \theta^\ell\big) + q^\ell, \quad \text{in } \Omega^\ell \times (0,T),, \tag{1.4}$$

Frictional contact problems for electro-viscoelastic materials with long-term memory, damage, and adhesion was studies in [9]. A dynamic contact problem between elasto-viscoplastic piezoelectric bodies can be found in [10]. A bilateral contact problem with adhesion and damage between two viscoelastic bodies was studies in [7]. A frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal effects can be found in [3].

The goal of this paper is to make the coupling of two thermo-viscoelastic bodies and a frictional contact problem with wear and damage. We study a quasistatic problem of frictional bilateral contact with wear and damage. We model the materiels behavior with an thermo viscoelastic constitutive law and the contact is frictional and bilateral.

This article is organized as follows. In Section 2 we describe the mathematical models for the frictional contact problem between two thermo-viscoelastics bodies with damage. The contact is modelled with normal compliance and wear. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. In Section 4 we state and prove our main existence and uniqueness result, the prove is carried out in several steps and is based on arguments of evolutionary variational inequalities, a classical existence and uniqueness result on parabolic inequalities, differential equations and the Banach fixed point theorem.

2 Problem Statement

Let us consider two thermo-viscoelastic bodies with wear and damage occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d(d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, such that $meas\Gamma_1^\ell > 0$. The two bodies are in bilateral frictional wear contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. Let $T > 0$ and let $[0, T]$ be the time interval of interest We admit a possible external heat source applied in $\Omega^\ell \times (0, T)$, given by the functions $\mathbf{q}^\ell$. The body Ω^ℓ is clamped on $\Gamma_1^\ell \times (0, T)$. The surface tractions $\mathbf{f}_2^\ell$ act on $\Gamma_2^\ell \times (0, T)$ and a body forces of density $\mathbf{f}_0^\ell$ acts on $\Omega^\ell \times (0, T)$.

We model the materials with thermo-viscoelastic constituve law with damage bodies. We also assume that the normal derivate of ζ^ℓ represent a homogeneous Neumann boundary conditions where

$$\frac{\partial \zeta^\ell}{\partial \nu^\ell} = 0 \text{ with } \zeta = \left(\zeta^1, \zeta^2\right)$$

The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. We introduce the wear function $\omega : \Gamma_3 \times (0, T) \to \mathbb{R}^+$ wich measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Let g be the initial gap between the two bodies and let p_ν and p_τ denote the normal and tangential campliance functions. We denote by $\mathbf{v}^*$ and $\alpha^* = \|\mathbf{v}^*\|$ the tangential velocity and the tangentiel speed at the contact surface between the two bodies. We use the modified version of Archard's law:

$$\dot{\omega} = -\kappa_\omega \mathbf{v}^* \boldsymbol{\sigma}_\nu$$

To describe the evolution of wear, where $\kappa_\omega > 0$ is a wear coefficient. We introduce the unitary vector $\delta : \Gamma_3 \to \mathbb{R}^d$ defined by $\delta = \mathbf{v}^*/\|\mathbf{v}^*\|$. When the contact arises, some material of the contact surfaces worn out and immediatly removed from the system. This process is measured by the wear function ω.

With the assumption above, the classical formulation of the friction contact problem with wear and damage between two thermo-viscoelastics bodies is following.

Problem P

For $\ell = 1, 2$, find a displacement field $\mathbf{u}^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times (0,T) \to \mathbb{S}^d$, a temperature $\theta^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}$, a damage $\zeta^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}$, and a wear $\omega : \Gamma_3 \times (0,T) \to \mathbb{R}^+$ such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell), \zeta^\ell\big) + \mathcal{F}^\ell(\theta^\ell, \zeta^\ell), \quad \text{in } \Omega^\ell \times (0,T), \tag{2.1}$$

$$\dot{\theta}^\ell - \kappa_0 \Delta\theta^\ell = \psi^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell), \theta^\ell\big) + \mathbf{q}^\ell, \quad \text{in } \Omega^\ell \times (0,T), \tag{2.2}$$

$$\dot{\zeta}^\ell - \kappa^\ell \Delta\zeta^\ell + \partial\psi_{K^\ell}(\zeta^\ell) \ni \phi^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell), \zeta^\ell\big) \quad \text{in } \Omega^\ell \times (0,T), \tag{2.3}$$

$$Div\boldsymbol{\sigma}^\ell + \mathbf{f}_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{2.4}$$

$$\mathbf{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{2.5}$$

$$\boldsymbol{\sigma}^\ell \cdot \nu^\ell = \mathbf{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{2.6}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}_\nu^1 = \boldsymbol{\sigma}_\nu^2 \equiv \boldsymbol{\sigma}_\nu, \\ \boldsymbol{\sigma}_\nu = p_\nu(\mathbf{u}_v - \omega - g), \end{aligned}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = -p_\tau(\mathbf{u}_v - \omega - g)\tfrac{\mathbf{v}^*}{\|\mathbf{v}^*\|}, \end{aligned}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.8}$$

$$\mathbf{u}_v^1 + \mathbf{u}_v^2 = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\dot{\omega} = -\kappa_\omega \alpha^* \boldsymbol{\sigma}_\nu = \kappa_\omega \alpha^* p_\nu(\mathbf{u}_v - \omega - g), \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$k_1 \frac{\partial\theta^\ell}{\partial\nu} + B^\ell\theta^\ell = 0, \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.11}$$

$$\frac{\partial\zeta^\ell}{\partial\nu^\ell} = 0, \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{2.12}$$

$$\mathbf{u}^\ell(0) = \mathbf{u}_0^\ell, \theta^\ell(0) = \theta_0^\ell, \quad \zeta^\ell(0) = \zeta_0^\ell, \quad \text{in } \Omega^\ell, \tag{2.13}$$

$$\omega(0) = \omega_0, \quad \text{on } \Gamma_3 \tag{2.14}$$

First, equations (2.1) and(2.2)represent the thermo-viscoelastic constitutive law with damage, the evolution of the damage is governed by the inclusion of parabolic type given by the relation (2.3). Equation (2.4) is the equilibrium equations for the stress. Equations (2.5) and (2.6) represent the displacement and traction boundary condition, respectively. Condition (2.7) and (2.8)represents the frictional bilateral contact with wear described above.Equation (2.9) means that the two bodies are inseparable.

Next, the equation (2.10) represents the ordinary differential equation which describes the evolution of the wear function. Equations (2.11) and (2.12) represent, respectively on Γ, a Fourier boundary condition for the temperature and an homogeneous Neumann boundary condition for the damage field on Γ. the functions $\mathbf{u}_0^\ell$, θ_0^ℓ, $\zeta_0^\ell$and ω_0 in (2.13) and (2.14) are the initial data.

3 Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\mathbf{u}^\ell.\mathbf{v}^\ell = u_i^\ell.v_i^\ell, \quad |\mathbf{v}^\ell| = (\mathbf{v}^\ell.\mathbf{v}^\ell)^{\frac{1}{2}}, \quad \forall \mathbf{u}^\ell, \mathbf{v}^\ell \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad |\boldsymbol{\tau}^\ell| = (\boldsymbol{\tau}^\ell \cdot \boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^\ell = \{\mathbf{v}^\ell = (v_i^\ell); v_i^\ell \in L^2(\Omega^\ell)\}, \quad H_1^\ell = \{\mathbf{v}^\ell = (v_i^\ell); v_i^\ell \in H^1(\Omega^\ell)\},$$
$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell); \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\}, \quad \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell; div\boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell.\mathbf{v}^\ell dx, \quad (\mathbf{u}^\ell, \mathbf{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell.\mathbf{v}^\ell dx + \int_{\Omega^\ell} \nabla\mathbf{u}^\ell.\nabla\mathbf{v}^\ell dx,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} Div\boldsymbol{\sigma}^\ell.div\boldsymbol{\tau}^\ell dx$$

and the associated norms $\|\cdot\|_{H^\ell}$, $\|\cdot\|_{H_1^\ell}$, $\|\cdot\|_{\mathcal{H}^\ell}$, and $\|\cdot\|_{\mathcal{H}_1^\ell}$ respectively. Here and below we use the notation

$$\nabla\mathbf{u}^\ell = (u_{i,j}^\ell), \quad \varepsilon(\mathbf{u}^\ell) = (\varepsilon_{ij}(\mathbf{u}^\ell)), \quad \varepsilon_{ij}(\mathbf{u}^\ell) = \frac{1}{2}(u_{i,j}^\ell + u_{j,i}^\ell), \quad \forall u^\ell \in H_1^\ell,$$
$$Div\boldsymbol{\sigma}^\ell = (\sigma_{ij,j}^\ell), \quad \forall \boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell.$$

For every element $\mathbf{v}^\ell \in H_1^\ell$, we also use the notation $\mathbf{v}^\ell$ for the trace of $\mathbf{v}^\ell$ on Γ^ℓ and we denote by v_ν^ℓ and $\mathbf{v}_\tau^\ell$ the *normal* and the *tangential* components of $\mathbf{v}^\ell$ on the boundary Γ^ℓ given by

$$v_\nu^\ell = \mathbf{v}^\ell.\nu^\ell, \quad \mathbf{v}_\tau^\ell = \mathbf{v}^\ell - v_\nu^\ell \boldsymbol{\nu}^\ell.$$

Let H'_{Γ^ℓ} be the dual of $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}$ denote the duality pairing between H'_{Γ^ℓ} and H_{Γ^ℓ}. For every element $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$ let $\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell$ be the element of H'_{Γ^ℓ} given by

$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell, \mathbf{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = (\boldsymbol{\sigma}^\ell, \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell} + (Div\boldsymbol{\sigma}^\ell, \mathbf{v}^\ell)_{H^\ell} \quad \forall \mathbf{v}^\ell \in H_1^\ell.$$

Denote by σ_ν^ℓ and $\boldsymbol{\sigma}_\tau^\ell$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$, respectively. If $\boldsymbol{\sigma}^\ell$ is continuously differentiable on $\Omega^\ell \cup \Gamma^\ell$, then

$$\sigma_\nu^\ell = (\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell) \cdot \boldsymbol{\nu}^\ell, \quad \boldsymbol{\sigma}_\tau^\ell = \boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell - \sigma_\nu^\ell\boldsymbol{\nu}^\ell,$$
$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell, \mathbf{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell \cdot \mathbf{v}^\ell da$$

for all $\mathbf{v}^\ell \in H_1^\ell$, where da is the surface measure element.

To obtain the variational formulation of the problem (2.1)–(2.14), we introduce for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \{\mathbf{v}^\ell \in H_1^\ell; \mathbf{v}^\ell = 0 \text{ on } \Gamma_1^\ell\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality holds:

$$\|\varepsilon(\mathbf{v}^\ell)\|_{\mathcal{H}^\ell} \geq c_K \|\mathbf{v}^\ell\|_{H_1^\ell} \quad \forall \mathbf{v}^\ell \in V^\ell, \tag{3.1}$$

Where the constant c_K denotes a positive constant which may depends only on Ω^ℓ, Γ_1^ℓ (see [13]). Over the space V^ℓ we consider the inner product given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{V^\ell} = (\varepsilon(\mathbf{u}^\ell), \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \mathbf{u}^\ell, \mathbf{v}^\ell \in V^\ell, \tag{3.2}$$

Let $\|\cdot\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (3.1) that the norms $\|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|\cdot\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.2), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\mathbf{v}^\ell\|_{L^2(\Gamma_3)^d} \leq c_0 \|\mathbf{v}^\ell\|_{V^\ell} \quad \forall \mathbf{v}^\ell \in V^\ell. \tag{3.3}$$

We also introduce the spaces

$$E_0^\ell = L^2(\Omega^\ell),$$
$$E_1^\ell = H^1(\Omega^\ell),$$

To simplify notation, we define the product spaces

$$\begin{gathered} \boldsymbol{V} = V^1 \times V^2, \\ H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \\ E_0 = E_0^1 \times E_0^2,\ E_1 = E_1^1 \times E_1^2, \end{gathered}$$

The spaces $\boldsymbol{V}$, E_1 are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_{\boldsymbol{V}}$, $(\cdot,\cdot)_{E_1}$. The associate norms will be denoted by $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{E_1}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$, $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\begin{gathered} \|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X, \\ \|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X, \end{gathered}$$

respectively.

We recall the following standard result for parabolic variational inequalities used in section 4 (see [1, p124]). Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V . We write

$$V \subset H \subset V'.$$

We say that the inclusions above define a Gelfand triple. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$, and $\|\cdot\|_{V'}$, the norms on the spaces V , H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V. Note that if $f \in H$ then

$$(f,\mathbf{v})_{V'\times V} = (f,\mathbf{v})_H\,, \forall \mathbf{v} \in H.$$

To solve the **problem P** , we use the following theorem.

Theorem 3.1. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V . Assume that $\mathfrak{a}(\cdot,\cdot) : V\times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$\mathfrak{a}(\mathbf{v},\mathbf{v}) + c_0 \|\mathbf{v}\|_H^2 \geqslant \zeta \|\mathbf{v}\|_V^2 \ ,\forall \mathbf{v} \in H$$

Then, for every $\mathbf{u}_0 \in K$ *and* $f \in L^2(0,T;H)$*, there exists a unique function* $\mathbf{u} \in H^1(0,T;H) \cap L^2(0,T;V)$ *such that* $\mathbf{u}(0) = \mathbf{u}_0$*,* $\mathbf{u}(t) \in K$ *for all* $t \in [0,T]$*, and for almost all* $t \in (0,T)$*,*

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{V' \times V} + \mathfrak{a}(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \geqslant (f(t), \mathbf{v} - \mathbf{u}(t))_H \ , \forall \mathbf{v} \in K,$$

Proof. The proof of this theorem can be found in [1, P 123-124] □

This theorem will be used in section 4 to obtain the existence and uniqueness result.

In the study of the Problem **P**, we consider the following assumptions:
The *viscosity function* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(3.4)
(a) There exists $L_{\mathcal{A}^\ell} > 0$ such that $|\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2)| \leq L_{\mathcal{A}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) There exists $m_{\mathcal{A}^\ell} > 0$ such that $(\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathcal{A}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(c) The mapping $\boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\xi})$ is Lebesgue measurable on Ω^ℓ,
for any $\boldsymbol{\xi} \in \mathbb{S}^d$.
(d) The mapping $\boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \mathbf{0})$ is continuous on $\mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.

The *elasticity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

(3.5)
(a) There exists $L_{\mathcal{G}^\ell} > 0$ such that $|\mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1, \boldsymbol{\zeta}_1) - \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2, \boldsymbol{\zeta}_2)| \leq L_{\mathcal{G}^\ell}\big(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\big)$, for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, for all $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) The mapping $\boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\xi} \in \mathbb{S}^d$,and for all $\boldsymbol{\zeta} \in \mathbb{R}$.
(c) The mapping $\boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x}, \mathbf{0}, \mathbf{0})$ belongs to $\mathcal{H}^\ell$.

The *thermal expansion operator* $\mathcal{F}^\ell : \Omega^\ell \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

(3.6)
(a) There exists $L_{\mathcal{F}^\ell} > 0$ such that $|\mathcal{F}^\ell(\boldsymbol{x}, \boldsymbol{\theta}_1, \boldsymbol{\zeta}_1) - \mathcal{F}^\ell(\boldsymbol{x}, \boldsymbol{\theta}_2, \boldsymbol{\zeta}_2)| \leq L_{\mathcal{F}^\ell}\big(|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2| + |\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\big)$, for all $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}$, for all $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) The mapping $\boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{\zeta})$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\theta}, \boldsymbol{\zeta} \in \mathbb{R}$.
(c) The mapping $\boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x}, \mathbf{0}, \mathbf{0})$ belongs to $\mathcal{H}^\ell$.

The *damage source function* $\phi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

(3.7)
(a) There exists $L_{\phi^\ell} > 0$ such that $|\phi^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1, \zeta_1) - \phi^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2, \zeta_2)| \leq L_{\phi^\ell}\big(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\zeta_1 - \zeta_2|\big)$, for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$ and $\zeta_1, \zeta_2 \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\ell$,
(b) The mapping $\boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x}, \boldsymbol{\xi}, \zeta)$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\xi} \in \mathbb{S}^d$ and $\zeta \in \mathbb{R}$,
(c) The mapping $\boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x}, \boldsymbol{0}, 0)$ belongs to $L^2(\Omega^\ell)$,
(d) $\phi^\ell(\boldsymbol{x}, \boldsymbol{\xi}, \zeta)$ is bounded for all $\boldsymbol{\xi} \in \mathbb{S}^d$, $\zeta \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\ell$.

The *nonlinear constitutive function* $\psi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

(3.8)
(a) There exists $L_{\psi^\ell} > 0$ such that $|\psi^\ell(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\xi}_1, \theta_1) - \psi^\ell(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\xi}_2, \theta_2)| \leq L_{\psi^\ell}\big(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\theta_1 - \theta_2|\big)$, for all $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$ and $\theta_1, \theta_2 \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\ell$,
(b) The mapping $\boldsymbol{x} \mapsto \psi^\ell(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\xi}, \theta)$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathbb{S}^d$ and $\theta \in \mathbb{R}$,
(c) The mapping $\boldsymbol{x} \mapsto \psi^\ell(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0)$ belongs to $L^2(\Omega^\ell)$,
(d) $\psi^\ell(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\xi}, \theta)$ is bounded for all $\boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathbb{S}^d$, $\theta \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\ell$.

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

(3.9)
(a) There exists $L_\nu > 0$ such that $|p_\nu(\boldsymbol{x}, u_1) - p_\nu(\boldsymbol{x}, u_2)| \leq L_\nu |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.
(b) $(p_\nu(\boldsymbol{x}, u_1) - p_\nu(\boldsymbol{x}, u_2))(u_1 - u_2) \geq 0$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.
(c) The mapping $\boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$.
(d) $p_\nu(\boldsymbol{x}, u) = 0$ for all $u \leq 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.

The *tangentiel contact function* $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

(3.10)
(a) There exists $L_\tau > 0$ such that $|p_\tau(\boldsymbol{x}, u_1) - p_\tau(\boldsymbol{x}, u_2)| \leq L_\tau |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) The mapping $\boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, u)$ is Lebesgue measurable. on Γ_3 for all $u \in \mathbb{R}$.
(c) The mapping $\boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, 0)$ belongs to $L^2(\Gamma_3)$.

We also suppose the following regularities

$$\text{(3.11)}\qquad \begin{gathered}\mathbf{f}_0^\ell \in L^2(0,T;L^2(\Omega^\ell)^d), \quad \mathbf{f}_2^\ell \in L^2(0,T;L^2(\Gamma_2^\ell)^d),\\ \mathbf{q} \in L^2(0,T;L^2(\Omega^\ell)),\end{gathered}$$

$$\text{(3.12)}\qquad \mathbf{u}_0^\ell \in V^\ell,$$

$$\text{(3.13)}\qquad \zeta_0^\ell \in K^\ell,$$

$$\text{(3.14)}\qquad \omega_0 \in L^2(\Gamma_3),$$

$$\text{(3.15)}\qquad p_\nu(.,u) \in L^2(\Gamma_3), p_\tau(.,u) \in L^2(\Gamma_3), u \in \mathbb{R},$$

$$\text{(3.16)}\qquad g \in L^2(\Gamma_3), \quad g \geq 0, \ \text{a.e. on } \Gamma_3$$

where K^ℓ is the set of admissible damage functions defined in (1.3).

Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0,T] \to \boldsymbol{V}$ as follows:

$$\text{(3.17)}\qquad (\mathbf{f}(t), \mathbf{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \cdot \mathbf{v}^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t) \cdot \mathbf{v}^\ell \, da \quad \forall \mathbf{v} \in \boldsymbol{V}$$

Next, we define the mappings $a : E_1 \times E_1 \to \mathbb{R}$, the wear functional $j : \boldsymbol{V} \times \boldsymbol{V} \times L^2(\Gamma_3) \to \mathbb{R}$, respectively by

$$\text{(3.18)}\qquad a(\zeta, \xi) = \textstyle\sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla\zeta^\ell \cdot \nabla\xi^\ell dx,$$

$$\text{(3.19)}\qquad \begin{aligned}&j(\mathbf{u}, \mathbf{v}, \omega) = \textstyle\int_{\Gamma_3} \Big(p_\nu \left(\mathbf{u}_v - \omega - g\right) \mathbf{v}_\nu\Big) da +\\ &\textstyle\int_{\Gamma_3} \Big(p_\tau \left(\mathbf{u}_v - \omega - g\right) \mathbf{v}_\nu\Big) \left\|\mathbf{v}_\tau - \mathbf{v}^*\right\| .da,\\ &\text{for all } \mathbf{u}, \mathbf{v} \in \boldsymbol{V}, \omega \in L^2(\Gamma_3).\end{aligned}$$

We note that conditions (3.14) imply

$$\text{(3.20)}\qquad \mathbf{f} \in L^2(0,T;\boldsymbol{V})$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical problem (2.1)–(2.14).

Problem PV

Find a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, a temperature $\theta = (\theta^1, \theta^2) : [0,T] \to V$, a damage $\zeta = (\zeta^1, \zeta^2) : [0,T] \to E_1$, a wear $\omega : [0,T] \to L^2(\Gamma_3)$ such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\varepsilon(\dot{\mathbf{u}}^\ell)+\mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell),\zeta^\ell\big) + \mathcal{F}^\ell(\theta^\ell,\zeta^\ell), \quad \text{in } \Omega^\ell \times (0,T), \tag{3.21}$$

$$\begin{aligned}&\sum_{\ell=1}^{2}(\boldsymbol{\sigma}^\ell, \varepsilon(\mathbf{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(t)))_{\mathcal{H}^\ell} + j(\mathbf{u}(t),\mathbf{v},\omega(t))\\ &- j(\mathbf{u}(t),\dot{\mathbf{u}}(t),\omega(t)) \geq (\mathbf{f}(t), \mathbf{v}-\dot{\mathbf{u}}(t))_{\boldsymbol{V}}\end{aligned} \tag{3.22}$$

$$\forall \mathbf{v} \in \boldsymbol{V}, \quad \text{a.e. } t \in (0,T),$$

$$\begin{aligned}&\sum_{\ell=1}^{2}(\dot{\theta}^\ell, \mathbf{v}) + a(\theta,\mathbf{v}) = \sum_{\ell=1}^{2}(\psi^\ell(\boldsymbol{\sigma}^\ell, \varepsilon(\mathbf{u}^\ell),\theta^\ell),\mathbf{v})\\ &+\sum_{\ell=1}^{2}(\mathbf{q}^\ell,\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{V},\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\sum_{\ell=1}^{2}(\dot{\zeta}^\ell(t), \xi^\ell - \zeta^\ell(t))_{L^2(\Omega^\ell)} + a(\zeta(t),\xi-\zeta(t))\\ &\geq \sum_{\ell=1}^{2}\Big(\phi^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell(t)),\zeta^\ell(t)\big), \xi^\ell - \zeta^\ell(t)\Big)_{L^2(\Omega^\ell)},\end{aligned} \tag{3.24}$$

$$\forall \xi \in K, \text{a.e. } t \in (0,T),$$

$$\dot{\omega} = \kappa_\omega \alpha^* p_\nu(\mathbf{u}_v - \omega - g) \tag{3.25}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \theta(0) = \theta_0, \zeta(0) = \zeta_0, \tag{3.26}$$

$$\omega(0) = \omega_0, \tag{3.27}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, a temperature, a damage, and a wear. The existence of the unique solution of problem **PV** is stated and proved in the next section.

4 Existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 4.1. *Assume that* (3.4)–(3.16) *hold. Then there exists a unique solution of Problem PV. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0,T;\boldsymbol{V}), \tag{4.1}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{4.2}$$

$$\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;V), \tag{4.3}$$

$$\zeta \in H^1(0,T;E_0) \cap L^2(0,T;E_1), \tag{4.4}$$

$$\omega \in C^1(0,T;L^2(\Gamma_3)), \tag{4.5}$$

The functions $\mathbf{u}$, $\boldsymbol{\sigma}$, θ, ζ and ω which satisfy (3.21)-(3.27) are called a weak solution of the contact Problem **P**. We conclude that, under the assumptions (3.4)– (3.16), the mechanical problem (2.1)–(2.14) has a unique weak solution satisfying (4.1)–(4.5).

The proof of Theorem 4.1 will be done in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and Banach fixed point theorem. To this end, we assume in what follows that (3.4)–(3.16) hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\mathcal{G}^\ell$, $\mathcal{F}^\ell$, ψ^ℓ, ϕ^ℓ, κ^ℓ, and T. but does not depend on t nor of the rest of input data, and whose value may change from place to place.

In order to prove the theorem, we consider for $\omega \in \mathcal{C}^1(0,T;\mathcal{L}^2(\Gamma_3))$; $\eta \in \mathcal{C}(0,T;\mathcal{H})$; $h \in \mathcal{C}(0,T;\boldsymbol{V})$; $\mu \in \mathcal{C}(0,T;V')$ and $\chi \in \mathcal{C}(0,T;\mathcal{L}^2(\Gamma_3))$, the following four auxiliary problems.

PROBLEM PV$_{\omega\eta h}$

Let $\mathbf{v}_{\omega\eta h} = \dot{\mathbf{u}}_{\omega\eta h}$ and $\mathbf{u}_{\omega\eta h} = \left(\mathbf{u}^1_{\omega\eta h}, \mathbf{u}^2_{\omega\eta h}\right)$

Find a displacement field $\mathbf{v}_{\omega\eta h} : [0,T] \to \boldsymbol{V}$, a stress field $\sigma_{\omega\eta h} : [0,T] \to \mathcal{H}$ such that

$$\sigma^\ell_{\omega\eta h} = \mathcal{A}^\ell \varepsilon(\mathbf{v}^\ell_{\omega\eta h}(t)) + \eta^\ell(t)$$

$$\sum_{\ell=1}^{2} (\sigma^\ell_{\omega\eta h}(t), \varepsilon(\mathbf{v}^\ell) - \varepsilon(\mathbf{v}^\ell_{\omega\eta h}(t)))_{\mathcal{H}^\ell} + j(h(t), \mathbf{v}, \omega(t)) - j(h(t), \mathbf{v}_{\omega\eta h}, \omega(t))$$
$$\geq (\mathbf{f}(t), \mathbf{v} - \mathbf{v}_{\omega\eta h}(t))_{\boldsymbol{V}} \quad \forall \mathbf{v} \in \boldsymbol{V},\ t \in (0,T),$$
$$\mathbf{u}_{\omega\eta h}(0) = \mathbf{u}_0$$

PROBLEM PV$\omega\mu$

Find the tempetrature $\theta_{\omega\mu} : [0,T] \to V$ wich is solution of the following variational problem

$$\textstyle\sum_{\ell=1}^{2} \left(\dot{\theta}^\ell_{\omega\mu}, \mathbf{v}\right) + a(\theta_{\omega\mu}, \mathbf{v}) = \sum_{\ell=1}^{2} (\mu^\ell(t) + \mathbf{q}^\ell(t), \mathbf{v}), \quad \forall \mathbf{v} \in V$$
$$\theta_\mu(0) = 0$$

PROBLEM PV$\omega\chi$

Find a damage $\zeta_{\omega\chi} = (\zeta^1_{\omega\chi}, \zeta^2_{\omega\chi}) : [0,T] \to H^1(\Omega)$ such that $\zeta_{\omega\chi}(t) \in K$ and

$$\begin{array}{l}\sum_{\ell=1}^{2}(\dot{\zeta}^\ell_{\omega\chi}(t), \xi^\ell - \zeta^\ell_{\omega\chi}(t))_{L^2(\Omega^\ell)} + a(\zeta_{\omega\chi}(t), \xi - \zeta_{\omega\chi}(t)) \\ \geq \sum_{\ell=1}^{2}\left(\chi^\ell(t), \xi^\ell - \zeta^\ell_{\omega\chi}(t)\right)_{L^2(\Omega^\ell)}, \quad \forall \xi \in K, \quad a.e. t \in (0,T), \\ \zeta_{\omega\chi}(0) = 0\end{array}$$

Where $K = K^1 \times K^2$.

PROBLEM PVω

Find a wear $\omega \in \mathcal{C}^1(0,T; \mathcal{L}^2(\Gamma_3))$ such that

$$\dot{\omega} = \kappa_\omega \alpha^* p_\nu(\mathbf{u}_v - \omega - g)$$
$$\omega(0) = \omega_0,$$

First step

Let $\omega \in \mathcal{C}^1(0,T; \mathcal{L}^2(\Gamma_3))$, $\eta \in \mathcal{C}(0,T;\mathcal{H})$and $h \in \mathcal{C}(0,T;V)$ we consider the following variational problem.

Problem PV$_{\omega\eta h}$

Let $\mathbf{v}_{\omega\eta h} = \dot{\mathbf{u}}_{\omega\eta h}$ and $\mathbf{u}_{\omega\eta h} = \left(\mathbf{u}^1_{\omega\eta h}, \mathbf{u}^2_{\omega\eta h}\right)$

Find a displacement field $\mathbf{v}_{\omega\eta h} : [0,T] \to \boldsymbol{V}$, a stress field $\sigma_{\omega\eta h} : [0,T] \to \mathcal{H}$ such that

$$\sigma^\ell_{\omega\eta h} = \mathcal{A}^\ell \varepsilon(\mathbf{v}^\ell_{\omega\eta h}(t)) + \eta^\ell(t) \tag{4.6}$$

(4.7)

$$\begin{array}{l}\displaystyle\sum_{\ell=1}^{2}(\sigma^\ell_{\omega\eta h}(t), \varepsilon(\mathbf{v}^\ell) - \varepsilon(\mathbf{v}^\ell_{\omega\eta h}(t)))_{\mathcal{H}^\ell} + j(h(t), \mathbf{v}, \omega(t)) - j(h(t), \mathbf{v}_{\omega\eta h}, \omega(t)) \\ \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{v}_{\omega\eta h}(t))_{\boldsymbol{V}} \quad \forall \mathbf{v} \in \boldsymbol{V},\ t \in (0,T), \\ \qquad\qquad \mathbf{u}_{\omega\eta h}(0) = \mathbf{u}_0\end{array}$$

We have the following result for the problem $\mathbf{PV}_{\omega\eta h}$.

Lemma 4.2. *$PV_{\omega\eta h}$ has a unique weak solution such that $\mathbf{v}_{\omega\eta h} \in C(0,T;\boldsymbol{V})$, and $\sigma_{\omega\eta h} \in C(0,T;\mathcal{H}_1)$ to the problem* (4.6) *and* (4.7)*.*

Proof. We define the operators $A : \boldsymbol{V} \to \boldsymbol{V}$ such that:

$$(A\mathbf{u}, \mathbf{v})_{\boldsymbol{V}} = \textstyle\sum_{\ell=1}^{2}(\mathcal{A}^{\ell}\boldsymbol{\varepsilon}(\mathbf{u}^{\ell}), \boldsymbol{\varepsilon}(\mathbf{v}^{\ell}))_{\mathcal{H}^{\ell}}, \tag{4.8}$$

It follows from (3.4)(a) and (4.8) that:

$$\|A\mathbf{u} - A\mathbf{v}\|_{\boldsymbol{V}} \le L_A \|\mathbf{u} - \mathbf{v}\|_{\boldsymbol{V}} \quad \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{V}. \tag{4.9}$$

Wich shows that $A : \boldsymbol{V} \to \boldsymbol{V}$ is Lipschitz continuous. Now by (3.4)(b) and (4.8) we find:

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_{\boldsymbol{V}} \ge m_A \|\mathbf{u} - \mathbf{v}\|_{\boldsymbol{V}}^2 \quad \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{V}. \tag{4.10}$$

Wich shows that $A : \boldsymbol{V} \to \boldsymbol{V}$ is a strongly monotone operator on $\boldsymbol{V}$**.**

Moreover, using Riesz representation theorem we may define the functions $\mathbf{F}_\eta : [0, T] \to \boldsymbol{V}$ by

$$\mathbf{F}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}(t) \quad \forall t \in [0, T],$$

Since A is a strongly monotone operator and Lipschitz continuous operator on $\boldsymbol{V}$ and since $j(h(t), \mathbf{v}, \omega(t))$is a proper convex lower semicontinuous functional, it follows from classical result on elliptic inequalities (see[8]) that there exists a unique function $\mathbf{v}_{\omega n h}(t) \in \boldsymbol{V}$ such that

$$\sum_{\ell=1}^{2}(\mathcal{A}^{\ell}\varepsilon(\mathbf{v}_{\omega\eta h}^{\ell}(t)), \varepsilon(\mathbf{v}^{\ell}) - \varepsilon(\mathbf{v}_{\omega\eta h}^{\ell}(t)))_{\mathcal{H}^{\ell}} + j(h(t), \mathbf{v}, \omega(t)) - j(h(t), \mathbf{v}_{\omega\eta h}(t), \omega(t))$$
$$\ge (\mathbf{F}_\eta(t), \mathbf{v} - \mathbf{v}_{\omega\eta h}(t))_{\boldsymbol{V}} \quad \forall \mathbf{v} \in \boldsymbol{V},\ t \in (0, T), \tag{4.11}$$

We use the relation (4.6), the assumptions (3.4), we obtain that

$$\sigma_{\omega n h}(t) \in \mathcal{H}$$

Using the definition (3.17) for $\mathbf{f}$, we deduce

$$Div \sigma_{\omega\eta h}(t) + \mathbf{f}_0(t) = 0 \tag{4.12}$$

With the regularity assumption (3.11) we see that

$$Div \sigma_{\omega\eta h}(t) \in H \text{ therfore } \sigma_{\omega\eta h}(t) \in \mathcal{H}_1$$

Let now $t_1,\, t_2 \in [0, T]$, and denote $\eta\,(t_i) = \eta_i$, $\mathbf{f}\,(t_i) = \mathbf{f}_i$, $h\,(t_i) = h_i$,

$\mathbf{v}_{\omega\eta h}\,(t_i) = \mathbf{v}_i$, $\sigma_{\omega\eta h}\,(t_i) = \sigma_i$, for $i = 1, 2$.Using the relation (4.11) we find that

$$\begin{aligned} &(A\mathbf{v}_1 - A\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)_{\boldsymbol{V}} \\ &\le (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{v}_1 - \mathbf{v}_2)_{\boldsymbol{V}} + (\eta_2 - \eta_1, \varepsilon\,(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}} \\ &+ j\,(h_1, \mathbf{v}_2, \omega) - j\,(h_1, \mathbf{v}_1, \omega) + j\,(h_2, \mathbf{v}_1, \omega) - j\,(h_2, \mathbf{v}_2, \omega) \end{aligned} \tag{4.13}$$

From the definition of the functional j given by (3.19) we have

$$\begin{aligned}
&j(h_1,\mathbf{v}_2,\omega)-j(h_1,\mathbf{v}_1,\omega)+j(h_2,\mathbf{v}_1,\omega)-j(h_2,\mathbf{v}_2,\omega)\\
&=\int_{\Gamma_3}\{p_\nu\left(h_{1\nu}-\omega-g\right)-p_\nu\left(h_{2\nu}-\omega-g\right)\}\left(v_{2\nu}-v_{1\nu}\right)da\\
&+\int_{\Gamma_3}\{p_\tau\left(h_{1\tau}-\omega-g\right)-p_\tau\left(h_{2\tau}-\omega-g\right)\}\left(\|v_{2\tau}-v^*\|-\|v_{1\tau}-v^*\|\right)da
\end{aligned}$$

We use (3.3), (3.9) and (3.10) to deduce that

$$\begin{aligned}
&j(h_1,\mathbf{v}_2,\omega)-j(h_1,\mathbf{v}_1,\omega)+j(h_2,\mathbf{v}_1,\omega)-j(h_2,\mathbf{v}_2,\omega)\\
&\leq C\|h_1-h_2\|_{\boldsymbol{V}}\|\mathbf{v}_1-\mathbf{v}_2\|_{\boldsymbol{V}}
\end{aligned} \tag{4.14}$$

The relation (3.2), the estimate (4.10) and the inequality (4.14) combined with (4.13) give us

$$m_A\left\|\mathbf{v}_1-\mathbf{v}_2\right\|_{\boldsymbol{V}}\leq C\left(\left\|\mathbf{f}_1-\mathbf{f}_2\right\|_V+\left\|\eta_1-\eta_2\right\|_{\mathcal{H}}+\left\|h_1-h_2\right\|_{\boldsymbol{V}}\right) \tag{4.15}$$

The inequality (4.15) and the regularity of the function $\mathbf{f}$, h and η show that

$$\mathbf{v}_{\omega\eta h}\in C(0,T;\boldsymbol{V})$$

From assumption (3.4) and the relation (4.6) we have

$$\left\|\sigma_1-\sigma_2\right\|_{\mathcal{H}}\leq C\left(\left\|\mathbf{v}_1-\mathbf{v}_2\right\|_V+\left\|\eta_1-\eta_2\right\|_{\mathcal{H}}\right) \tag{4.16}$$

and from (4.12) we have.

$$Div\sigma(t_i)+\mathbf{f}_0(t)=0,\ \ i=1,2. \tag{4.17}$$

The regularity of the function η, $\mathbf{v}$, $\mathbf{f}_0$ and the relation (4.16)-(4.17)show that

$$\sigma_{\omega\eta h}\in C(0,T;\mathcal{H}_1)$$

Let $\omega\in C(0,T;\boldsymbol{L}^2\left(\Gamma_3\right)), h\in C(0,T;\boldsymbol{V})$ and let $\eta\in C(0,T;\mathcal{H})$ be given. We consider the following operator

$$\Lambda_{\omega\eta}:C\left(0,T;\boldsymbol{V}\right)\rightarrow C\left(0,T;\boldsymbol{V}\right)$$

Defined by

$$\Lambda_{\omega\eta}h=\mathbf{u}_0+\int_0^t\mathbf{v}_{\omega\eta h}(s)ds\quad\forall h\in C\left(0,T;\boldsymbol{V}\right) \tag{4.18}$$

□

Lemma 4.3. *Let the assumptions (3.4)-(3.16) hold. Then the operator $\Lambda_{\omega\eta}$ has a unique fixed point $h_{\omega\eta} \in C(0,T;\boldsymbol{V})$,*

Proof. Let $h_1, h_2 \in C(0,T;V)$ and let $\eta \in C(0,T;\mathcal{H})$,we use the relation $\mathbf{v}_{\omega\eta h_i} = \mathbf{v}_i$and $\boldsymbol{\sigma}_{\omega\eta h_i} = \sigma_i$ for $i = 1,2$.

Using similar arguments as those used in (4.15) we find that

$$m_{\mathcal{A}} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_{\boldsymbol{V}} \leq C \|h_1(t) - h_2(t)\|_{\boldsymbol{V}} \quad \forall t \in [0,T] \tag{4.19}$$

From (4.18)(a) and (4.19) we have

$$\|\Lambda_{\omega\eta}h_1 - \Lambda_{\omega\eta}h_2\|_{\boldsymbol{V}} \leq C \int_0^t \|h_1(t) - h_2(t)\|_{\boldsymbol{V}}\, ds \quad \forall t \in [0,T] \tag{4.20}$$

Reitrating this inequality m times, we obtain

$$\|\Lambda_{\omega\eta}h_1 - \Lambda_{\omega\eta}h_2\|_{C(0,T;\boldsymbol{V})} \leq \frac{C^m T^m}{m!} \|h_1 - h_2\|_{C(0,T;\boldsymbol{V})}\, ds \quad \forall t \in [0,T]$$

This show that for m large enough the operator $\Lambda^m_{\omega\eta}$ is a contraction in the Banach space. Thus, from Banach's fixed point theorem the operator $\Lambda_{\omega\eta}$ has a unique fixed point $h^*_{\omega\eta} \in C(0,T;\boldsymbol{V})$. □

For $\eta \in \mathcal{C}(0,T;\mathcal{H})$, let $h^*_{\omega\eta}$ be the fixed point given by the above lemma, i.e. $h_{\omega\eta^*} = \mathbf{v}_{\omega\eta^* h}$. In the sequel we denote by $(\mathbf{v}_{\omega\eta}, \boldsymbol{\sigma}_{\omega\eta}) \in \mathcal{C}(0,T;\boldsymbol{V}) \times \mathcal{C}(0,T;\mathcal{H}_1)$ the unique solution of Problem $\mathbf{PV}_{\omega\eta h}$, i.e. $\mathbf{v}_{\omega\eta} = \mathbf{v}_{\omega\eta^* h}$, $\sigma_{\omega\eta} = \sigma_{\omega\eta^* h}$. Also, we denote by $\mathbf{u}_{\omega\eta} : [0,T] \to \boldsymbol{V}$ the function defined by

$$\mathbf{u}_{\omega\eta}(t) = \int_0^t \mathbf{v}(s)ds + \mathbf{u}_0, \quad \forall t \in [0,T]. \tag{4.21}$$

From Lemma 4.2 we deduce that

$$\mathbf{u}_{\omega\eta} \in C^1(0,T;\boldsymbol{V})$$

Now we consider the following problem

Problem PV$\omega\eta$

Find a displacement field $\mathbf{u}_{\omega\eta} : [0,T] \to \boldsymbol{V}$ such that for all $t \in [0,T]$

(4.22)

$$\sum_{\ell=1}^{2} (A^\ell \varepsilon(\dot{\mathbf{u}}^\ell_{\omega\eta}(t)), \varepsilon(\mathbf{v}^\ell) - \varepsilon(\dot{\mathbf{u}}^\ell_{\omega\eta}(t)))_{\mathcal{H}^\ell}$$
$$+ j(\mathbf{u}^\ell_{\omega\eta}(t), \mathbf{v}, \omega(t)) - j(\mathbf{u}^\ell_{\omega\eta}(t), \dot{\mathbf{u}}^\ell_{\omega\eta}(t), \omega(t)) + (\eta(t)^\ell, \varepsilon(\mathbf{v}^\ell) - \varepsilon(\dot{\mathbf{u}}^\ell_{\omega\eta}(t)))_{\mathcal{H}^\ell}$$
$$\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}^\ell_{\omega\eta}(t))_{\boldsymbol{V}} \quad \forall \mathbf{v} \in \boldsymbol{V},\ t \in (0,T),$$

$$\mathbf{u}_{\omega\eta}(0) = \mathbf{u}_0 \tag{4.23}$$

We have the following result for the problem $\mathbf{PV}_{\omega\eta}$.

Lemma 4.4. *$PV_{\omega\eta}$ has a unique weak solution satisfying the regularity* (4.1).

Proof. For each $\omega \in C(0,T;L^2(\Gamma_3))$ and $\eta \in C(0,T;\mathcal{H})$,we denote by $h_{\omega\eta} \in C(0,T;\boldsymbol{V})$ the fixed point obtained in Lemma 4.3 and let $\mathbf{u}_{\omega\eta}$ be the function defined by

$$\mathbf{u}_{\omega\eta}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\omega\eta h_{\omega\eta}}(s)ds \quad \forall t \in [0,T] \tag{4.24}$$

We have $\Lambda_{\omega\eta}h_{\omega\eta} = h_{\omega\eta}$ from (4.18) and (4.24) it follows that

$$\mathbf{u}_{\omega\eta} = h_{\omega\eta} \tag{4.25}$$

Therefore, taking $h = h_{\omega\eta}$ in (4.7) and using (4.6),(4.24) and (4.25) we see that $\mathbf{u}_{\omega\eta}$ is the unique solution to the problem $\mathbf{PV}_{\omega\eta}$ satisfiying the regularity expressed in (4.1) . □

Second step

In the second step, we use the displacement field $\mathbf{u}_{\omega\eta}$ obtained in Lemma 4.4 and Let $\mu \in \mathcal{C}(0,T;V')$, we consider the following variational problem.

Problem PV$\omega\mu$

Find the tempetrature $\theta_{\omega\mu} : [0,T] \to V$ wich is solution of the following variational problem

$$\begin{gathered}\sum_{\ell=1}^2 \left(\dot{\theta}^{\ell}_{\omega\mu}, \mathbf{v}\right) + a(\theta_{\omega\mu}, \mathbf{v}) = \sum_{\ell=1}^2 (\mu^{\ell}(t) + \mathbf{q}^{\ell}(t), \mathbf{v}), \quad \forall \mathbf{v} \in V \\ \theta_{\mu}(0) = 0\end{gathered} \tag{4.26}$$

We have the following result.

Lemma 4.5. *$PV_{\omega\mu}$ has a unique solution $\theta_{\omega\mu}$ which satisfies the regularity* (4.3).

Proof. By an application of the Friedrichs-Poincaré inequality, we can find a constant $\beta' > 0$ such that

(4.27)

$$\sum_{\ell=1}^2 \int_{\Omega^\ell} \|\xi\|^2 dx + \frac{\beta}{k_0} \sum_{\ell=1}^2 \int_{\Gamma^\ell} \|\xi\|^2 d\gamma \geq \beta' \sum_{\ell=1}^2 \int_{\Omega^\ell} \|\xi\|^2 dx, \quad \forall \xi \in \boldsymbol{V}.$$

Thus, we obtain

$$a_0(\xi, \xi) \geq c_1 \sum_{\ell=1}^2 \|\xi\|_V^2, \quad \forall \xi \in \boldsymbol{V}. \tag{4.28}$$

Where $c_1 = k_0 \min(1, \beta')/2$, wich implies that a_0 is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (4.26) has a unique solution $\theta_{\omega\mu}$ satisfiying (4.3). □

Third step

In the third step, we use the displacement field $\mathbf{u}_{\omega\eta}$ obtained in Lemma 4.4 and we consider the following initial-value problem.

Problem PV$\omega\chi$

Find a damage $\zeta_{\omega\chi} = (\zeta^1_{\omega\chi}, \zeta^2_{\omega\chi}) : [0, T] \to H^1(\Omega)$ such that $\zeta_{\omega\chi}(t) \in K$ and

$$\begin{aligned}\sum_{\ell=1}^{2}(\dot{\zeta}^\ell_{\omega\chi}(t), \xi^\ell - \zeta^\ell_{\omega\chi}(t))_{L^2(\Omega^\ell)} + a(\zeta_{\omega\chi}(t), \xi - \zeta_{\omega\chi}(t)) \\ \geq \sum_{\ell=1}^{2}\left(\chi^\ell(t), \xi^\ell - \zeta^\ell_{\omega\chi}(t)\right)_{L^2(\Omega^\ell)}, \quad \forall \xi \in K, \quad a.e. t \in (0, T),\end{aligned} \tag{4.29}$$

$$\zeta_{\omega\chi}(0) = 0 \tag{4.30}$$

Where $K = K^1 \times K^2$, to solve problem $\mathbf{PV}\omega\chi$, we recall the following abstract result for parabolic variational inequalities,

Lemma 4.6. *There exists a unique solution* $\zeta_{\omega\chi}$ *of Problem* $\mathrm{PV}_{\omega\chi}$ *and it satisfies*

$$\zeta_{\omega\chi} \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

Proof. Using (3.18), (3.21) and a classical existence and uniqueness result on parabolic equations (see [1, P 124]) □

Fourth step

In the fourth step, we use the displacement field $\mathbf{u}_{\omega\eta}$ obtained in Lemma 4.4 and we consider the following initial-value problem.

Problem PVω

Find a wear $\omega \in \mathcal{C}^1(0, T; \mathcal{L}^2(\Gamma_3))$ such that

$$\dot{\omega} = \kappa_\omega \alpha^* p_\nu(\mathbf{u}_v - \omega - g) \tag{4.31}$$

$$\omega(0) = \omega_0, \tag{4.32}$$

Let us now we consider the operator $\mathcal{L} : \mathcal{C}(0,T;L^2(\Gamma_3)) \to \mathcal{C}(0,T;L^2(\Gamma_3))$ defined by

$$\mathcal{L}\omega\,(t) = -k_1\mathbf{v}^* \int_0^t (\sigma_\omega)_V\,(s)\,ds \quad \forall t \in [0,T]. \tag{4.33}$$

Lemma 4.7. *The operator $\mathcal{L}$ has a unique fixed point ω^* and it satisfies*

$$\omega^* \in C(0,T;L^2(\Gamma_3))$$

Proof. Let $\omega_1, \omega_2 \in \mathcal{C}(0,T;L^2(\Gamma_3))$, and $t \in [0,T]$. We denote by $(\mathbf{u}_i, \boldsymbol{\sigma}_i, \theta_i, \zeta_i)$, for $i = 1,2$ the solution to the problem $\mathbf{PV}_\omega$ for $\omega = \omega_i$ use the notation $\mathbf{u}_{\omega_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\omega_i} = \mathbf{v}_{\omega_i} = \mathbf{v}_i$, $\zeta_{\omega_i} = \zeta_i$, $\theta_{\omega_i} = \theta_i$ and $\sigma_{\omega_i} = \sigma_i$, where $\mathbf{u}_i = (\mathbf{u}_i^1, \mathbf{u}_i^2)\,, \zeta_i = (\zeta_i^1, \zeta_i^2)$. Morever we denote in sequel by C various positive constants wich may depend on $k_1$and $\mathbf{v}^*$. Using similar arguments that those used in the proof of the relation (4.39), to find that

$$\begin{aligned}&\textstyle\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\,ds\\ &\le C\big(\textstyle\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\,ds + \int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2\,ds\big)\end{aligned} \tag{4.34}$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$ and using (4.34) we obtain

$$\begin{aligned}&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \le C\textstyle\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2\,ds\\ &\le C\big(\textstyle\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\,ds + C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2\,ds\big)\end{aligned} \tag{4.35}$$

Applying Gronwall inequality, we deduce that

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \le C\textstyle\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2\,ds \tag{4.36}$$

So, by (4.34), (4.36) , it follows that

$$\textstyle\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\,ds \le C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2\,ds \tag{4.37}$$

On other hand since

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\varepsilon(\dot{\mathbf{u}}_i{}^\ell) + \mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}_i^\ell), \zeta_i^\ell\big) + \mathcal{F}^\ell(\theta^\ell, \zeta_i^\ell) \tag{4.38}$$

For $i = 1,2$ we use the assumption (3.4)(b), (3.5), (3.6) and(3.7) to obtain for $s \in [0,T]$

$$\|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 \le C\big(\|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\big) \tag{4.39}$$

We integrate the previous inequality with respect to time to deduce that

$$\begin{aligned}&\textstyle\int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2\,ds\\ &\le C\big(\textstyle\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\,ds + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\,ds\big)\end{aligned} \tag{4.40}$$

We substitute (4.36) and (4.37) in the previous inequality to find

$$(4.41) \qquad \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 \, ds \le C \int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 \, ds$$

The definition of the operator $\mathcal{L}$ given by (4.33) and estimate (4.37) give us

$$(4.42) \qquad \|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)\|_{L^2(\Gamma_3)}^2 \le C \int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Gamma_3)}^2 \, ds$$

Reiterating this inequality n times leads to

$$(4.43) \qquad \|\mathcal{L}^n\omega_1 - \mathcal{L}^n\omega_2\|_{\mathcal{C}(0,T;L^2(\Gamma_3))}^2 \le \tfrac{C^nT^n}{n!} \|\omega_1(s) - \omega_2(s)\|_{\mathcal{C}(0,T;L^2(\Gamma_3))}^2$$

□

Therfore,for n large enough, $\mathcal{L}^n$ is contractive operator on the Banach space $\mathcal{C}(0,T;L^2(\Gamma_3))$. The operator $\mathcal{L}$ has a unique fixed point $\omega^* \in \mathcal{C}(0,T;L^2(\Gamma_3))$.

Now we have all the ingredients to prove Theorem4.1

Proof of theorem. By taking into account the above results and the properties of the operators $\mathcal{G}^\ell$ and $\mathcal{F}^\ell$ and of the functions ψ^ℓ and ϕ^ℓ, we may consider the operator $\Lambda : \mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega)) \to \mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$ □

$$(4.44) \qquad \Lambda(\eta,\mu,\chi)(t) = (\Lambda_1(\eta,\mu,\chi)(t), \Lambda_2(\eta,\mu,\chi)(t), \Lambda_3(\eta,\mu,\chi)(t)),$$

defined by

$$(4.45) \quad \Lambda_1(\eta,\mu,\chi)(t) = \sum_{\ell=1}^{2} \mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}_{\omega\eta}^\ell), \zeta_{\omega\chi}^\ell\big) + \mathcal{F}^\ell(\theta_{\omega\mu}^\ell, \zeta_{\omega\chi}^\ell),$$

$$(4.46) \quad \Lambda_2(\eta,\mu,\chi)(t) = \big(\psi^1\left(\sigma_{\omega\eta}^1, \varepsilon(\dot{\mathbf{u}}_{\omega\eta}^1), \theta_{\omega\mu}^1\right), \psi^2\left(\boldsymbol{\sigma}_{\omega\eta}^2, \varepsilon(\dot{\mathbf{u}}_{\omega\eta}^2), \theta_{\omega\mu}^2\right)\big),$$

$$(4.47) \quad \Lambda_3(\eta,\mu,\chi)(t) = \big(\phi^1\left(\boldsymbol{\varepsilon}(\mathbf{u}_{\omega\eta}^1), \zeta_{\omega\chi}^1\right), \phi^2\left(\boldsymbol{\varepsilon}(\mathbf{u}_{\omega\eta}^2), \zeta_{\omega\chi}^2\right)\big),$$

Here for every $(\eta,\mu,\chi) \in \mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$, $\mathbf{u}_\eta$, θ_μ, ζ_χ and ω representes the displacement , the temperature, the damage and the wear obtained in Lemma4.4, Lemma4.5, Lemma4.6 and Lemma4.7 respectivly and

$$(4.48) \qquad \boldsymbol{\sigma}_\omega^\ell = \mathcal{A}^\ell\varepsilon(\dot{\mathbf{u}}_\omega^\ell) + \mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}_\omega^\ell), \zeta_\omega^\ell\big) + \mathcal{F}^\ell(\theta_\omega^\ell, \zeta_\omega^\ell)$$

We have the following result.

Lemma 4.8. *Let (4.3) be satisfied. Then for* $(\boldsymbol{\eta},\mu,\chi)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$ *, the maping* $\Lambda(\boldsymbol{\eta},\mu,\chi):[0,T]\to\mathcal{H}\times V'\times L^2(\Omega)$ *has a unique element* $(\boldsymbol{\eta}^*,\mu^*,\chi^*)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$ *such that* $\Lambda(\boldsymbol{\eta}^*,\mu^*,\chi^*)=(\boldsymbol{\eta}^*,\mu^*,\chi^*)$.

Proof. Let $(\boldsymbol{\eta}_1,\mu_1,\chi_1),(\boldsymbol{\eta}_2,\mu_2,\chi_2)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$, and $t\in[0,T]$.

We use the notation $\mathbf{u}_{\omega\eta_i}=\mathbf{u}_i,\dot{\mathbf{u}}_{\omega\eta_i}=\mathbf{v}_{\omega\eta_i}=\mathbf{v}_i,\zeta_{\omega\chi_i}=\zeta_i,\theta_{\omega\mu_i}=\theta_i$ and $\boldsymbol{\sigma}_{\omega\eta_i}=\boldsymbol{\sigma}_i$,for $i=1,2$.

Using (3.2) and the relations (3.5)-(3.7), we obtain

(4.49)
$$\begin{aligned}&\|\Lambda(\eta_1,\mu_1,\chi_1)(t)-\Lambda(\eta_2,\mu_2,\chi_2)(t)\|_{\mathcal{H}\times V'\times L^2(\Omega)}\\ &\le L_{\mathcal{G}}\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}\Big)\\ &+L_{\mathcal{F}}\int_0^t\Big(\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}+L_{\mathcal{A}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V\\ &+\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V+\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\Big)ds\\ &+M_\phi\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}\Big)\\ &+L_\psi\Big(\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}+\|\mathbf{v}_1(t)-\mathbf{v}_2(t)\|_V+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}\Big)\end{aligned}$$

Since

$$\mathbf{u}_i\,(t)=\int_0^t\mathbf{v}_i\,(s)\,ds+\mathbf{u}_0,\forall t\in[0,T]\,,\tag{4.50}$$

we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\le\int_0^t\|\mathbf{v}_1\,(s)-\mathbf{v}_2\,(s)\,\|_V ds,\tag{4.51}$$

Applying Young's and Hölder's inequalities, (4.49) becomes, via (4.51) ,

$$\begin{aligned}&\|\Lambda(\eta_1,\mu_1,\chi_1)(t)-\Lambda(\eta_1,\mu_1,\chi_1)(t)\|_{\mathcal{H}\times V'\times L^2(\Omega)}\\ &\le C\Big(\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Omega)}+\int_0^t\big(\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\\ &+\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V+\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\\ &+\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\big)ds\Big).\end{aligned}\tag{4.52}$$

Taking in mind that

$$\boldsymbol{\sigma}_i^\ell(t)=\mathcal{A}^\ell(\varepsilon(\mathbf{v}_i(t)))+\boldsymbol{\eta}_i^\ell\,(t)\,,\forall t\in[0,T]\,.\tag{4.53}$$

it follows

$$\begin{aligned}&\textstyle\sum_{\ell=1}^2(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell\,(t)),\boldsymbol{\varepsilon}(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell\,(t))_{\mathcal{H}^\ell}\\ &\le j(\mathbf{v}_1,\mathbf{v}_2,\omega)+j(\mathbf{v}_2,\mathbf{v}_1,\omega)-j(\mathbf{v}_1,\mathbf{v}_1,\omega)-j(\mathbf{v}_2,\mathbf{v}_2,\omega)\end{aligned}\tag{4.54}$$

So, by using (3.4), (3.19) and (3.3), we deduce that

(4.55)
$$m_{\mathcal{A}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2 \leq C_0^2\|\alpha\|_{L^\infty(\Gamma_3)}\Big(\|\lambda\|_{L^\infty(\Gamma_3)}+1\Big)\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2 + \|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2$$

Which, by the Gronwall inequality, implies

$$\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2 \leq C\|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}}^2 \tag{4.56}$$

Then

$$\begin{aligned}\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V ds &\leq C\int_0^t\int_0^s \|\eta_1(r)-\eta_2(r)\|_{\mathcal{H}} dr ds\\ &\leq \int_0^T \|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}} ds\end{aligned} \tag{4.57}$$

For the temperature , if we take the substitution $\mu=\mu_1$, $\mu=\mu_2$ in (4.26) and subtracting the two obtained equations, we deduce by choosing $\mathbf{v}=\theta_1-\theta_2$ as test function

(4.58)
$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2 + C_1\int_0^t \|\theta_1(t)-\theta_2(t)\|_V^2 \leqslant \int_0^t \|\mu_1(s)-\mu_2(s)\|_{V'}\|\theta_1(s)-\theta_2(s)\|_V ds,\ \forall t\in[0,T],$$

Employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}&\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\theta_1(s)-\theta_2(s)\|_V^2 ds\\ &\qquad\leqslant C\int_0^t \|\mu_1(s)-\mu_2(s)\|_{V'}^2 ds\ ,\forall t\in[0,T].\end{aligned} \tag{4.59}$$

We use the inclusion $L^2(\Omega)\subset V$, we get

$$\begin{aligned}&\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2 ds\\ &\qquad\leqslant C\int_0^t \|\mu_1(s)-\mu_2(s)\|_{V'}^2 ds\ ,\forall t\in[0,T].\end{aligned} \tag{4.60}$$

From this inequality, combined with Gronwall's inequality, we deduce that

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2 \leqslant C\textstyle\int_0^t \|\mu_1(s)-\mu_2(s)\|_V^2 ds \tag{4.61}$$

For the damage field, from (4.29) we deduce that

(4.62)
$$(\dot{\zeta}_1-\dot{\zeta}_2,\zeta_1-\zeta_2)_{L^2(\Omega)}+\mathfrak{a}_1(\zeta_1-\zeta_2,\zeta_1-\zeta_2)\leqslant(\chi_1-\chi_2,\zeta_1-\zeta_2)_{L^2(\Omega)},\quad a.e\ \ t\in(0,T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\zeta_1(0) = \zeta_2(0) = \zeta_0$ and inequality $\mathfrak{a}_1(\zeta_1 - \zeta_2, \zeta_1 - \zeta_2) \geqslant 0$ to find

$$\frac{1}{2}\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Omega)} \leqslant \int_0^t (\chi_1(s) - \chi_2(s), \zeta_1(s) - \zeta_2(s))_{L^2(\Omega)} ds, \tag{4.63}$$

which implies

$$\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Omega)} \leqslant \int_0^t \|\chi_1(s) - \chi_2(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|\zeta_1(s) - \zeta_2(s))\|^2_{L^2(\Omega)} ds. \tag{4.64}$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Omega)} \leqslant C \int_0^t \|\chi_1(s) - \chi_2(s)\|^2_{L^2(\Omega)} ds, \quad \forall t \in [0, T]. \tag{4.65}$$

Applying the previous inequalities, the estimates (4.61) and (4.65), we substitute(4.52) to obtain

$$\begin{aligned} &\|\Lambda(\eta_1, \mu_1, \chi_1)(t) - \Lambda(\eta_2, \mu_2, \chi_2)(t)\|^2_{V \times L^2(\Omega)} \\ &\qquad \leqslant C \int_0^T \|(\eta_1, \mu_1, \chi_1)(s) - (\eta_2, \mu_2, \chi_2)(s)\|^2_{V \times L^2(\Omega)} ds. \end{aligned} \tag{4.66}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathcal{C}(0, T; V \times L^2(\Omega))$, and so Λ has a unique fixed point in this Banach space. □

Existence. Let $(\boldsymbol{\eta}^*, \mu^*, \chi^*) \in \mathcal{C}(0, T; \mathcal{H} \times V' \times L^2(\Omega))$ be the fixed point of Λ defined by (4.44)-(4.47) and let $h^* = h^*_{\eta^*}$ be the fixed point of the operator Λ_{η^*} given by Lemma4.2. We denote

$$\begin{aligned} \mathbf{u}_* &= \mathbf{u}_{\omega\eta^*}, \theta_* = \theta_{\omega\mu^*}, \zeta_* = \zeta_{\omega\chi^*}. \\ \boldsymbol{\sigma}^\ell_* &= \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell_*) + \mathcal{G}^\ell\big(\boldsymbol{\varepsilon}(\mathbf{u}^\ell_*), \zeta^\ell_*\big) + \mathcal{F}^\ell(\theta^\ell_*, \zeta^\ell_*) \end{aligned}$$

$\Lambda_1(\boldsymbol{\eta}^*, \mu^*, \chi^*) = \boldsymbol{\eta}^*$, $\Lambda_2(\boldsymbol{\eta}^*, \mu^*, \chi^*) = \mu^*$ and $\Lambda_3(\boldsymbol{\eta}^*, \mu^*, \chi^*) = \chi^*$, the definitions (4.45)-(4.47) show that (3.21)-(3.27) are satisfied. Next, from Lemmas 4.2, 4.4, 4.5, 4.6and 4.7, the regularity conditions (4.1)-(4.5) follow. □

Uniqueness. Let ω^* be the fixed point of the operator $\mathcal{L}$ given by (4.33). The unique solution $(\mathbf{u}_*, \boldsymbol{\sigma}_*, \theta_*, \zeta_*, \omega^*)$ is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.44)-(4.47) and the unique solvability of the Problem $\mathbf{PV}_{\omega\boldsymbol{\eta}h}$, $\mathbf{PV}_{\omega\mu}$, $\mathbf{PV}_{\omega\chi}$and , $\mathbf{PV}_{\omega}$ which completes the proof. □

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[Upper Bounds For The Number of Limit Cycles for a Class of Generalized Liénard Polynomial Differential Systems Via The Averaging Methods]

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Abstract: One of the main problems in the theory of ordinary differential equations is the study of their limit cycles, their existence, their number and their stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. The second part of the 16th Hilbert's problem is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree.

In this work, we study the number of limit cycles of polynomial differential systems of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(h_1(x)\, y^{2\alpha} + g_1(x)\, y^{2\alpha+1} + f_1(x)\, y^{2\alpha+2}) \\ \qquad - \varepsilon^2(h_2(x)\, y^{2\alpha} + g_2(x)\, y^{2\alpha+1} + f_2(x)\, y^{2\alpha+2}) \end{cases}$$

where m, n, k and α are positive integers, h_i, g_i and f_i have degree n, m and k, respectively for each $i = 1, 2$, and ε is a small parameter. We provide an accurate upper bound of the maximum number of limit cycles that this class of systems can have bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of first and second order.

Keywords : limit cycles, averaging theory, Liénard differential systems.

A MIXED FORMULATION OF FLEXURAL PRESTRESSED SHELL MODEL

Abstract. In this work we present a finite element method for a prestressed shell model, to perform conforming method, we use a mixed method for the problem in Cartesian coordinate. We establish existence and uniqueness of the solution to the both formulations.

1. Introduction

A mixed formulation is a way for imposing a constraint a variational problems. for proving that the mixed formulation and the discret mixed formulation its well-posedness we demonstrat the bilinear form $b(\cdot,\cdot)$ satisfies the Inf-Sup condition. One reason behind opting for the mixed formulation is that the flexural model is among the models which suffer from the locking phenomena, while mixed formulation resolve this problem. As a second reason, the condition number of the penalized problem matrix is very large. We refer to [2] the first work using the mixed method for the shell in Cartesian coordinate. In this work we present a mixed formulation of a prestressed shell model presented in [1].

2. The constrained continuous problem.

Following to [1] the model takes the following variational form :

$$\begin{cases} \quad \text{Find } U=(u,r)\in\mathbb{V} \text{ such that} \\ \mathbf{a}(U,V)+\boldsymbol{a}_p(U,V)=\mathcal{L}(v), \quad \forall V=(v,s)\in\mathbb{V} \end{cases} \tag{1}$$

where

$$\mathbf{a}(U,V)=ta_m(u,v)+ta_t((u,r),(v,s))+\frac{t^3}{12}a_f(r,s), \quad \boldsymbol{a}_p(r,s)=\frac{t^3}{12}a_p(r,s) \text{ and } \mathcal{L}(v)=\int_\omega f\cdot v.$$

$$\mathbb{V}=\{(v,s)\in H^1(\omega,\mathbb{R}^3)\times L^2(\omega,\mathbb{R}^3): s\cdot a_\alpha\in H^1(\omega,\mathbb{R}), s\cdot a_3=\tilde{\gamma}_{12}(v), \quad v|_{\Gamma_0}=0\} \tag{2}$$

The bilinear forms a_m, a_t , a_f, and a_p correspond to the membrane, transverse shear, flexural and prestress effects, respectively [1]. The thickness t of the shell is assumed to be constant and positive.

3. A mixed formulation for problem (1).

In this section, we present a mixed formulation for the prestressed model (1). Let us consider the relaxed function space:

$$\mathbb{X}(\omega)=\{(v,s)\in H^1(\omega,\mathbb{R}^3)\times L^2(\omega,\mathbb{R}^3): s\cdot a_\alpha\in H^1(\omega,\mathbb{R}), \quad v|_{\Gamma_0}=0\} \tag{3}$$

equipped with the natural norm. and we set

$$\mathbb{M}(\omega)=L^2(\omega). \tag{4}$$

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Key words and phrases. Prestressed shell, finite element method, mixed formulation.

We consider the following variational problem:

$$\begin{cases} \text{Find } (U,\psi) = (u,r,\psi) \in \mathbb{X} \times \mathbb{M}(\omega) \text{ such that} \\ \mathbf{a}(U,V) + \mathbf{a}_p(U,V) + b(V,\psi) = \mathcal{L}(V), \forall V \in \mathbb{X}. \\ b(U,\phi) = 0, \quad \forall \phi \in \mathbb{M}(\omega) \end{cases} \tag{5}$$

where, $b(U,V) = \int_\omega (r \cdot a_3 - \tilde{\gamma}_{12}(u))(s \cdot a_3 - \tilde{\gamma}_{12}(v))\, dx$

Theorem 3.1. *There exists a constant $C > 0$ such that*

$$\forall \phi \in \mathbb{M}(\omega) \qquad \sup_{V \in \mathbb{X}(\omega)} \frac{b(V,\phi)}{\|V\|_{\mathbb{X}(\omega)}} \geq C\|\phi\|_{L^2(\omega)} \tag{6}$$

Theorem 3.2. *If $\|\nabla a_3\|_{L^\infty}$ is sufficiently small, the problem (5) has a unique solution (U,ψ), such that U is the solution of the problem (1).*

4. Finite element approximation

Let $(\mathcal{T}_h)_{h>0}$ be a regular affine family of triangulations which covers the domain ω. We introduce the finite dimensional spaces

$$\mathbb{X}_h = \{V_h = (v_h, s_h) \in (C^0(\bar{\omega}, \mathbb{R}^3))^2 / V_{h|T} \in \mathbb{P}_2(T,\mathbb{R}^3) \times \mathbb{P}_1(T,\mathbb{R}^3),\ \forall T \in \mathcal{T}_h\}. \tag{7}$$

$$\mathbb{M}_h = \{\mu_h \in C^0(\bar{\omega}) / \mu_{h|T} \in \mathbb{P}_1(T),\ \forall T \in \mathcal{T}_h\}. \tag{8}$$

we consider the following discrete problem: Then we consider the following discrete problem:

$$\begin{cases} \text{Find } (U_h,\psi_h) = (u_h,r_h,\psi_h) \in \mathbb{X}_h \times \mathbb{M}_h \text{ such that} \\ \mathbf{a}(U_h,V_h) + \mathbf{a}_p(U_h,V_h) + b(V_h,\psi_h) = \mathcal{L}(V_h), \forall V_h \in \mathbb{X}_h. \\ b(U_h,\phi_h) = 0, \quad \forall \phi_h \in \mathbb{M}_h \end{cases} \tag{9}$$

Proposition 4.1. *The discrete problem (9) has a unique solution.*

Proof. The existence and uniqueness of a solution to (9) is based on the discrete infsup condition given in Lemma(4.2). □

Lemma 4.2. *There exists $\beta_h > 0$ dependent of h such that*

$$\inf_{\mu_h \in \mathbb{M}_h} \sup_{V_h \in \mathbb{X}_h} \frac{b(V_h,\mu_h)}{\|V_h\|_{\mathbb{X}(\omega)}\|\mu_h\|_{L^2(\omega)}} \geq \beta_h \tag{10}$$

Lemma 4.3. *Let (U,ψ) be a solution of the problem (5) and (U_h,ψ_h) be a solution of the problem (9) then this following estimate is hold*

$$\|U - U_h\|_{\mathbb{X}} \leq c_{1h} \inf_{V_h \in \mathbb{X}_h} \|U - V_h\|_{\mathbb{X}} + c_2 \inf_{\phi_h \in \mathbb{M}_h} \|\psi - \phi_h\|_{\mathbb{M}}. \tag{11}$$

$$\|\psi - \psi_h\|_{\mathbb{M}} \leq c_{3h} \inf_{V_h \in \mathbb{X}_h} \|U - V_h\|_{\mathbb{X}} + c_{4h} \inf_{\phi_h \in \mathbb{M}_h} \|\psi - \phi_h\|_{\mathbb{M}}. \tag{12}$$

Such that c_{1h}, c_{3h} and c_{4h} dependent on $1/\beta_h$ and c_2 independent on h.

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Dyanmic contact problem frictional with wear for an elastic-viscoplastic body and damage

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Work plan

Work plan

Work plan

Work plan

Introduction and position of the problem

Introduction and position of the problem

We consider a class of evolutionary variational problems which describes the a dynamic contact problem for an elastic-viscoplastic body with wear and damage between a body and a conductive obstacle. The formulation is in a form of a system involving hyperbolic quasi-variational inequality (the displacement field) and parabolic inequalities (damage fields).
We prove the existence of a unique weak solution to the problem. The proof is based on arguments of hyperbolic quasi-variational inequalities, parabolic inequalities, differential equations and fixed point.

The abstract weak formulation is given by the following problem

$$\begin{cases} (\ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} + (A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} + (B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} \\ \quad + \left(\int_0^t G(\boldsymbol{\sigma}(s) - A\dot{\mathbf{u}}(t), \mathbf{u}(s), \alpha(s)) ds, \mathbf{v} - \dot{\mathbf{u}}(t) \right)_{V' \times V} + j(\dot{\mathbf{u}}(t), \mathbf{v}) \\ \quad + j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (f(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} \quad \text{a.e. } t \in (0, T) \end{cases}$$

$$\begin{cases} (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ \geq (S(\boldsymbol{\sigma} - A(\dot{\mathbf{u}}(t)), \mathbf{u}(t), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)} \\ \quad \xi \in K, \text{ a.e. } t \in (0, T) \end{cases}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0 \tag{1}$$

Here V , H and K are respectively spaces of admissible displacements, of stress and of damage, there are Hilbert spaces, K be a nonempty, closed, and convex set of V . defined by

$$K = \{\xi \in V \quad : \quad 0 \leq \xi(x) \leq 1 \text{ a.e. } x \in \Omega\}$$

The operators A, B and G respectively related to the behavior low of an elasticviscoplastic constitutive law material with damage. The operators A, B and G are defined on V. The functionals j and S are respectively determined by the conditions of contact friction with wear on part Γ3 and related to the source function of the damage and friction on part Γ_3. The data f is related to the traction forces and to the body forces. The functions $u_0, \dot{u}_0$ and α_0 gives the initial data of displacement, initial data of velocity and the initial data of damage.
We denote by $\mathbf{u}$ the displacement field, by $\boldsymbol{\sigma}$ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linear strain tensor. We use an elastic-viscoplastic constitutive law with damage. Here $[0, T]$ is the interval of the observation. The dot above u and α denotes the derivative of the displacement $\mathbf{u}$ and denotes the derivative of damage α with respect to the variable t.

In order to solve Problem 2.1, we impose the following assumptions We list the assumptions on the problems data. We assume that operators $A, B : V \to V', G : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to V'$ a functional $j : V \times V \to \mathbb{R}$, the damage source function $S : \mathcal{H} \times \mathcal{H} \times H^1(\Omega) \to \mathbb{R}$ and two initial values $u_0 \in V$, $V_0 \in H$ and $\xi_0 \in K$ such that
(H_1) There exists a constant $M_A > 0$ such that

$$(Au_1 - Au_2, u_1 - u_2) \geq M_A \|u_1 - u_2\|^2 \ , \forall u_1, u_2 \in V. \tag{2}$$

(H_2) There exists a constant $L_A > 0$ such that

$$\|Au_1 - Au_2\|_{V'} \leq L_A \|u_1 - u_2\|_V \ , \forall u_1, u_2 \in V. \tag{3}$$

(H_3) $B \in L(V', V)$ is strongly monotone, i.e., there exists a constant $M_B \succ 0$ such that

$$(Bu, u) \geq M_B \|u\|_V^2 \, , \, \forall u \in V. \tag{4}$$

(H_4) The norm of B is L_B, i.e.

$$\|Bu\|_{V'} \leq L_B \|u\|_V \, , \forall u \in V. \tag{5}$$

(H_5) For any $u, v \in V$

$$(Bu, v)_{V' \times V} = (Bv, u)_{V' \times V}. \tag{6}$$

for more details see [**?**]5]

(H_6) There exists a constant $L_G > 0$ such that

$$\begin{gathered} \| G(\sigma, \mathbf{u}, \alpha)\|_{V'} \leq L_A \left(\|\sigma\|_V + \|\mathbf{u}\|_V + \|\alpha\|_V\right) \\ \forall \sigma, \varepsilon \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R} \end{gathered} \tag{7}$$

(H_7) There exists a constant $L_j > 0$ such that

$$j(g_1, v_2) + j(g_2, v_1) - j(g_1, v_1) - j(g_2, v_2) \leq L_j \|g_1 - g_2\|_V \|v_1 - v_2\|_V$$

$$\forall g_1, g_2, v_1, v_2 \in V.(8)$$

(H_8) There exists a constant $C_j > 0$ such that

$$\begin{cases} j(g, v_1) - j(g, v_2) \leq C_j \|g\|_V \|v_1 - v_2\|_V, & \forall g_1, g_2, v_1, v_2 \in V \\ j(g_1, v) - j(g_2, v) \leq C_j \|g_1 - g_2\|_V \|g\|_V, & \end{cases} \tag{9}$$

(H_9) For any $u \in V$.

$$j(u, .) \text{ is a convex functional in } V\text{for all} \in V. \tag{10}$$

(H_{10}) The f function satisfies

$$f \in H^2(0, T; V'). \tag{11}$$

(H_{11}) There exists $M_S > 0$ such that

$$\|S(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\alpha})\| \leq M_S(\|\boldsymbol{\sigma}\| + \|\mathbf{u}\| + \|\alpha\|) \tag{12}$$

(H_{12}) The function S satisfies

$$S \in L^2(0, T; L^2(\Omega)). \tag{13}$$

We begin by study the following hyperbolic quasi-variational inequality:

Problem

Find $u \in L^2(0, T; V)$ *with* $\dot{u} \in L^2(0, T; V)$ *and* $\ddot{u} \in L^2(0, T; V)$ *such that*

$$\begin{aligned}&\langle \ddot{u}(t), v - \dot{u}(t)\rangle_{V' \times V} + \langle A(\dot{u}(t)), v - \dot{u}(t)\rangle_{V' \times V} + \langle B(u(t)), v - \dot{u}(t)\rangle_{V' \times V}\\ &+j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V' \times V}, \forall v \in V, \quad a.e.t \in [0, T]\end{aligned} \tag{14}$$

$$u(0) = u_0, \dot{u}_0 = v_0. \tag{15}$$

Theorem

If conditions (2)-(11) are satisfied and $L_j < M_A < 2C_j$. Then

$$u(t) \in C(0,T;V) \text{ with } \begin{cases} \dot{u}(t) \in C(0,T;V) \cap L^\infty(0,T;V) \\ \text{and} \\ \ddot{u}(t) \in C(0,T;V) \cap L^\infty(0,T;V') \end{cases} \tag{16}$$

where u is the unique solution of Problem 1.
*The proof of the above theorem 2, we use an abstract existence and unique result which may be found in [**?**]5]*

We begin by study the following parabolic variational inequalities

Problem

Find $\alpha(t) \in K$ such that

$$\langle \dot{\alpha}(t), \zeta - \alpha(t) \rangle_{V' \times V} + a(\dot{\alpha}(t), \zeta - \alpha(t)) \geq (\phi(t), \zeta - \alpha(t))_{L^2(\Omega)} \quad (17)$$

$$\alpha(0) = \alpha_0. \quad (18)$$

Theorem

Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(.,.) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants λ and γ,

$$a(\alpha, \alpha) + \gamma \|\alpha\|_H^2 \geq \lambda \|\alpha\|_V^2, \quad \forall \alpha \in V$$

Then, for every $\alpha_0 \in K$ and $S \in L^2(0, T; H)$, there exists a unique function $\alpha \in H^1(0, T; H) \cap L^2(0, T; V)$, such that $\alpha(0) = \alpha_0$, $\alpha(t) \in K$ for all $t \in [0, T]$, where α is the unique solution of Problem 3.

The existence of the unique solution of Problem $\mathcal{P}$ is the topic of the

Proof of the main result

Proof of the main result

Now, we propose our existence and uniqueness result.

Theorem

Assume that (H_1)-(H_{12}) *hold. Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \alpha)$ *to problem P1. Moreover, the solution satisfies*

$$\begin{cases} \mathbf{u}(t) \in C(0,T;V) \\ \dot{\mathbf{u}}(t), \in C(0,T;V) \cap L^\infty(0,T;V) \\ \ddot{\mathbf{u}}(t) \in C(0,T;V) \cap L^\infty(0,T;V') \end{cases} \tag{19}$$

$$\boldsymbol{\sigma} \in H^2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in H^2(0,T;V') \tag{20}$$

$$\alpha \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;V). \tag{21}$$

Then Problem p has a unique solution $(\mathbf{u}, \boldsymbol{\sigma}, \alpha)$ *hold (18), (19) and (20). The proof of theorem 5, is carried out is several steps and is based on the following abstract result for evolutionary hyperbolic quasi-variational inequality*

Let $\eta \in H^2(0, T; V')$ and $\theta \in H^2(0, T; L^2(\Omega))$ be given and consider the following variational problems

Problem

p_η *Find a displacement field* $u_\eta : [0, T] \to V$*, such that*

$$\left\{ \begin{array}{c} (\ddot{\mathbf{u}}_\eta(t), v - \dot{\mathbf{u}}_\eta(t))_{V' \times V} + (A\dot{\mathbf{u}}_\eta(t), v - \dot{\mathbf{u}}_\eta(t))_{V' \times V} \\ + (B\mathbf{u}_\eta(t), v - \dot{\mathbf{u}}_\eta(t))_{V' \times V} + (\boldsymbol{\eta}(t), v - \dot{\mathbf{u}}_\eta(t))_{V' \times V} + j(\dot{\mathbf{u}}_\eta(t), v) \\ - j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \geq (\mathbf{f}, v - \dot{\mathbf{u}}_\eta(t))_{V' \times V} \\ a.e. \ \ t \in (0, T), \qquad for\ all\ v \in V \end{array} \right. \tag{22}$$

$$\mathbf{u}_\eta(0) = u_0, \quad \dot{\mathbf{u}}_\eta(0) = \dot{u}_0 \tag{23}$$

Lemma

There exists u_η *a unique solution to Problem* P_η *and it has the regularity expressed in (19).*

Problem

p_θ *Find the damage field* $\alpha_\theta : [0, T] \to \mathbb{R}$

$$\alpha_\theta(t) \in K, (\dot{\alpha}_\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} + a(\alpha_\theta(t), \xi - \alpha_\theta(t)) \tag{24}$$

$$\geq (\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)}, \forall \xi \in K, \ a.e.t \ \in (0, T), \tag{25}$$

$$\alpha_\theta(0) = \alpha_0. \tag{26}$$

in studying the problem of $\mathcal{P}_\theta$ we have the following result

Lemma

problem $\mathcal{P}_\theta$ *has a unique solution* α_θ *such that*

$$\alpha_\theta \in W^{1,2}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right). \tag{27}$$

To solve $\mathcal{P}_\theta$, we recall the following standard result for parabolic variational inequalities (see, e.g.,[**?**]).

In the third step, we use u_η, α_θ obtained in Lemmas 7,11 respectively to construct the following Cauchy problem for the stress field.

Problem

Find the stress field $\sigma_{\eta,\theta} : [0, T] \to V$ *solution of the problem*

$$\boldsymbol{\sigma}_{\eta,\theta}(t) = B(\mathbf{u}_\eta(t)) + \int_0^t G(\boldsymbol{\sigma}_{\eta,\theta}(s), \mathbf{u}_\eta(s), \alpha_\theta(s))\, ds, \ \ a.e. \ \ t \in (0, T) \tag{28}$$

Lemma

There exists a unique solution $\boldsymbol{\sigma}_{\eta,\theta}$ of the problem $P_{\eta,\theta}$ and it satisfies(19). Moreover, if $\mathbf{u}_{\eta_i}$, α_{θ_i} and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$ represent the solutions to problems $\mathcal{P}_\eta$, $\mathcal{P}_\theta$ and $\mathcal{P}_{\eta,\sigma}$ for $i = 1, 2$, respectively, then there exists $C \succ 0$ such that

$$\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|^2_{\mathcal{H}} \leq C(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|^2_V + \int_0^t \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|^2_V ds + \int_0^t \|\alpha_{\theta_1}(s) - \alpha_{\theta_2}(s)\|^2_{L^2(\Omega)} ds). \tag{29}$$

Finally, following these results and using the properties of the operators A, B and F and the function S for $t \in (0, T)$, we consider the operator

$$\Lambda(\eta, \theta)(t) = \left(\Lambda^1(\eta, \theta)(t), \Lambda^2(\eta, \theta)(t)\right) \in V' \times L^2(\Omega) \tag{30}$$

defined by for all $vinV$

$$\begin{aligned} &\left(\Lambda^1(\eta, \theta)(t), v\right)_{V' \times V} = \left(\int_0^t G(\boldsymbol{\sigma}_{\eta,\theta}(s), \mathbf{u}_\eta(s), \alpha_\theta(s))\, ds, \mathbf{v}\right)_{V' \times V} \\ &\Lambda^2(\eta, \theta)(t) = \phi\left((\sigma_{\eta,\theta}(t), \mathbf{u}_\eta(t), \alpha_\theta(t)\right) \end{aligned} \tag{31}$$

Either we consider the mapping

$$\Lambda : H^2\left(0, T; V' \times L^2(\Omega)\right) \to H^2\left(0, T; V' \times L^2(\Omega)\right) \tag{32}$$

Lemma

The mapping Λ admits a fixed point $(\eta^*, \theta^*) \in H^2(0, T; V)$ such that

$$\Lambda(\eta^*, \theta^*) = (\eta^*, \theta^*) \tag{33}$$

Now let is go to the proof of Theorem 5
Let $(\eta^*, \theta^*) \in H^2(0, T; V)$, be the fixed point of Λ defined by (31)-(32) and denote

$$\mathbf{u} = \mathbf{u}_{\eta^*}, \quad \sigma = A(\dot{\mathbf{u}}) + \sigma_{\eta^*,\theta^*} \tag{34}$$

$$\alpha = \alpha_{\theta^*} \tag{35}$$

We prove that $(\mathbf{u}, \boldsymbol{\sigma}; \alpha)$ satisfies (1) and (19),(21). Indeed, we write (29) for$\eta^* = \eta, \theta^* = \theta$ and use (34), (35) to obtain

$$\boldsymbol{\sigma}(t) = A(\dot{\mathbf{u}}(t)) + B(\mathbf{u}(t)) + \int_0^T G(\boldsymbol{\sigma}(s) - A(\dot{\mathbf{u}}(s)), \mathbf{u}(s), \alpha(s))ds$$

Now we consider (22) for $\eta^* = \eta$ and use the first equality in (34) to obtain that (34) is satisfied.
The equalities $\Lambda^1(\eta^*, \theta^*) = \eta^*$ and $\Lambda^2(\eta^*, \theta^*) = \theta^*$ combined with (31), (34) and (35) show that for all $v \in V$,

$$(\eta^*(t), v)_{V' \times V} = \left(\int_0^t G(\sigma(s) - A(\mathbf{u}(s)), \mathbf{u}(s), \alpha(s))ds, v \right)_{V' \times V} \tag{36}$$

$$\theta^*(t) = \phi((\sigma(t) - A(\mathbf{u}(t)), \mathbf{u}(t), \alpha(t)) \tag{37}$$

We now substitute (34) in ((36) and (22) is satisfied. We write (24) for (35) in (37) and (24) is satisfied.
Next (1), and regularities (19) and (21), follow Lemmas 7, and 11. The regularity $\boldsymbol{\sigma} \in H^2(0, T; V')$ follows from Lemmas 7 and 12, the second equality in (34), (2) and ((3). Finally we obtain that $Div\boldsymbol{\sigma} \in H^2(0, T; V')$. We deduce that the regularity (20) holds which concludes the existence part of the theorem.

The uniqueness part of theorem 5 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (31) and the unique solvability of the Problem P_{η},P_{θ} and $P_{\eta,\theta}$ which completes the proof.

Application

We consider the following classical formulation of the mechanical dynamic contact problem friction with wear for an elastic-viscopalastic body and damage.

Find a displacement field $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, the damage field $\alpha : \Omega \times [0, T] \to \mathbb{R}$,

Problem

P

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) \\
&\quad + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon\left(u(s)\right), \alpha(s)\right) ds \qquad in\ \Omega \times (0,T), \qquad (38)
\end{aligned}$$

$$\rho\ddot{\mathbf{u}} = \operatorname{Div}\boldsymbol{\sigma} + f_0, \quad in\ \Omega \times (0,T), \qquad (39)$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni S(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(\mathbf{u}), \alpha), \quad in\ \Omega \times (0,T), \qquad (40)$$

$$\mathbf{u} = 0, \qquad on\ \Gamma_1 \times (0,T), \qquad (41)$$

$$\sigma\nu = f_2, \qquad on\ \Gamma_2 \times (0,T), \qquad (42)$$

$$\begin{cases} -\sigma_\nu = \beta\|\dot{u}_\nu\|, \|\boldsymbol{\sigma}_\tau\| = -\mu\sigma_\nu, \\ \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - v^*), \quad \lambda > 0, \end{cases} \qquad on\ \Gamma_3 \times (0,T), \qquad (43)$$

$$\frac{\partial\alpha}{\partial\nu} = 0, \qquad on\ \Gamma_3 \times (0,T), \qquad (44)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0, \quad \alpha(0) = \alpha_0, \qquad in\ \Omega. \qquad (45)$$

We now describe the notations in (38)-(45) and provide some comments on the equalities and the boundary conditions. First, equations (38) represent the elastic-viscoplastic law of a material constitutive law with damage, where $\mathcal{A}, \mathcal{B}$ and $\mathcal{G}$ are nonlinear operators describing the purely viscous, the elastic properties of the material and the visco-plastic behavior of the material, respectively. The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (39) where S is the mechanical source of the damage growth, assumed to be rather general function of the strains and damage itself, $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Equation (40) represents the equation of motion where f_0 is the density of the volume forces acting on the deformable body and ρ denotes the density of the mass. The conditions (41) and (42) are the displacement and the boundary conditions of traction, (43) represents the conditions of contact with wear and friction on part Γ3 . (44) describes a homogeneous Neumann boundary condition where $\dfrac{\partial\alpha}{\partial\nu}$ is the normal derivative of α. In, condition (45) gives the initial displacement, the initial velocity and α_0 is the initial damage.

We now list the assumptions on the problem's data.
The viscosity operator $\mathcal{A}:\ \Omega \times \mathbb{S}_d \longrightarrow \mathbb{S}_d$ satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \qquad\qquad \|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\| \leq L_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_d, x \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ \qquad\qquad \|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\|(\varepsilon_1, \varepsilon_2) \geq m_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\|^2 \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_d. \\ (c) \text{ The mapping } \ x \longrightarrow \mathcal{A}(x,\varepsilon) \text{ is lebesgue measurable on } \Omega, \\ \qquad\qquad \text{for all } \varepsilon \in \mathbb{S}_d. \\ (d) \text{ The mapping } \ x \longrightarrow \mathcal{A}(x,0) \in \mathcal{H}. \end{cases} \tag{46}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \mathcal{B}(\cdot) \in \mathcal{L}\left(\mathbb{S}^d, \mathbb{S}^d\right), \text{ i.e., there exists } \mathrm{L}_{\mathcal{B}} \text{ such that } \|\mathcal{B}(\varepsilon)\big) \| \leq \mathrm{L}_{\mathcal{B}} \| \varepsilon\|, \\ \qquad\qquad \text{for all } \varepsilon \in \mathbb{S}^d. \\ \text{There exists } m_{\mathcal{B}} \text{ such that} \\ \|\left(\mathcal{B}\left(x, \varepsilon_1\right) - \mathcal{B}\left(x, \varepsilon_2\right)\right)\| \left(\varepsilon_1 - \varepsilon_2\right) \geq \mathrm{m}_{\mathcal{B}} \left\|\varepsilon_1 - \varepsilon_2\right\|^2 \\ \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ \mathcal{B}\varepsilon_1 : \varepsilon_2 = \mathcal{B}\varepsilon_2 : \varepsilon_1 \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ and } \mathcal{B}(0) \in \mathcal{H}. \end{cases} \tag{47}$$

The relaxation function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$, satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{G}} > 0 \quad \text{such that} \\ \|\mathcal{G}(x, \sigma_1, \varepsilon_1, \alpha_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2, \alpha_2)\| \leq \\ \qquad\qquad L_{\mathcal{G}}(\|\sigma_1 - \sigma_2\| + |\varepsilon_1 - \varepsilon_2\| + \|\alpha_1 - \alpha_2\|) \\ \qquad\qquad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \forall t \in [0, T], x \in \Omega. \\ (b) \text{ The mapping } x \longrightarrow \mathcal{G}(x, \sigma, \varepsilon, \alpha) \text{ is lebesgue measurable on } \Omega, \\ \text{ for all } \varepsilon \in \mathbb{S}_d, \quad \forall t \in [0, T], \alpha \in \mathbb{R} \\ (c) \text{ The mapping } x \longrightarrow \mathcal{G}(x, 0, 0, 0) \in \mathcal{H}, \forall t \in [0, T]. \end{cases} \tag{48}$$

The function of the source of damages $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$, is satisfies

$$\begin{cases} (a) \text{ There exists } M_S > 0 \quad \text{such that} \\ \qquad \|S(x, \boldsymbol{\sigma}_1, \varepsilon_1, \alpha_1) - S(x, \boldsymbol{\sigma}_1, \varepsilon_2, \alpha_2)\| \leq M_S(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \\ \qquad \|\varepsilon_1 - \varepsilon_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, x \in \Omega. \\ (b) \text{ The mapping } x \longrightarrow S(x, \boldsymbol{\sigma}, \varepsilon, \alpha) \text{ is lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\sigma}, \varepsilon \in \mathbb{S}^d, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } x \longrightarrow S(x, 0, 0, 0) \in L^2(\Omega). \end{cases} \tag{49}$$

The body force f_0 and surface traction f_2 coefficient of friction μ and adhesion field β,mass density ρ and initial conditions u_0, v_0 have the following properties

$$\begin{aligned}
&\mathbf{f}_0 \in H^2\left(0, T; L^2(\Omega)^d\right), \quad \mathbf{f}_2 \in H^2\left(0, T; L^2\left(\Gamma_2\right)^d\right). \\
&\mu \in L^\infty\left(\Gamma_3\right), \quad \mu(x) \geq 0 \text{ for a.e. } x \in \Gamma_3. \\
&\beta \in L^\infty\left(\Gamma_3\right), \quad \beta(x) \geq \beta^* > 0 \text{ for a.e. } x \in \Gamma_3. \\
&\rho \in L^\infty(\Omega), \quad \rho(x) \geq \rho^* > 0 \text{ for a.e. } x \in \Omega. \\
&u_0 \in V, v_0 \in H, \alpha_0 \in K.
\end{aligned} \tag{50}$$

Here, $1 \leq p \leq +1$. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$

$$a(\xi, \zeta) = k \int_\Omega \nabla\xi\nabla\zeta dx. \tag{51}$$

and the microcrack diffusion coefficient verifies $k > 0$. For the field of displacements we need the closed subspace V of H_1 by

$$V = \{u \in H_1 / u = 0 \text{ on } \Gamma_1\}. \tag{52}$$

Next, we use the Riesz's representation theorem causes the existence of an element $f\colon [0, T] \to V'$, such that

$$(\mathbf{f}(t), v)_{V' \times V} = (\mathbf{f}_0(t), v)_H + (\mathbf{f}_2(t), v)_{L^2(\Gamma_2)^d} \quad \forall v \in V, \text{ a.e. } \mathbf{t} \in (0, T). \tag{53}$$

Note that conditions (50) and (53) implies that

$$\mathbf{f} \in H^2(0, T; V). \tag{54}$$

Next, we denote by $j : V \times V \to \mathbb{R}$, the application defined by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \beta \|u_\nu\| (\mu \|\mathbf{v}_\tau - \mathbf{v}^*\| + v_\nu)\, da \tag{55}$$

Theorem

Let assumptions (46)-(54) hold. Suppose that there exists a constant $\lambda_0 > 0$ depending on Γ_3, such that

$$\|\beta\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) < \lambda_0 \tag{56}$$

and

$$c_\gamma^2 \|\beta\|_{L^\infty(\Gamma_3)^d} \left(\|\mu\|_{L^\infty(\Gamma_3)^d} + 1\right) < m_{\mathcal{A}} < 2c_\gamma^2 \|\beta\|_{L^\infty(\Gamma_3)^d} \left(\|\mu\|_{L^\infty(\Gamma_3)^d} + 1\right), \tag{57}$$

where c_γ is a constant. Then Problem P has a unique solution $(\mathbf{u}, \boldsymbol{\sigma}, \alpha)$ hold (19),(20) and (21)

Using standard arguments, we obtain the variational formulation of (38)-(45)

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{F}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon\left(\mathbf{u}(s)\right), \alpha(s)\right) ds, \tag{58}$$

$$\begin{aligned}(\rho\ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{\mathcal{H}} + (\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t))\\ \geq (\boldsymbol{f}, (\mathbf{v} - \dot{\mathbf{u}}(t)))_{V' \times V}, \quad \forall \mathbf{v} \in V,\end{aligned} \tag{59}$$

$$\begin{aligned}\alpha(t) \in K, \quad (\dot{\alpha}(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t))\\ \geq (S(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \varepsilon(\mathbf{u}(t)), \alpha(t)), \zeta - \alpha(t))_{L^2(\Omega)}, \forall \zeta \in K,\end{aligned} \tag{60}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0. \tag{61}$$

Proof.

The proof is based on Theorem 2.2. Let V denote the space defined in (??) and $H = L^2(\Omega)^d$. Then there are an evolution triple of spaces $V \hookrightarrow H \hookrightarrow V'$ and a compactly embedding operator $i : V \hookrightarrow H$. Define two operators A, B and $G : V \to V$ with $m_{\mathcal{A}} = M_A$, and $L_{\mathcal{A}} = L_A$, $m_{\mathcal{B}} = M_B$, and $L_{\mathcal{B}} = L_B$ and $m_{\mathcal{G}} = M_G$, and $L_{\mathcal{G}} = L_G$, an inner product $((,))_H$ and a functional $f : V \to V'$ by setting

$$\begin{cases} ((\mathbf{u}, \mathbf{v}))_H = (\rho(x)\mathbf{u}, \mathbf{v})_H \\ \langle A\mathbf{u}, \mathbf{v} \rangle_{V' \times V} = (\mathcal{A}(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \\ \langle B\mathbf{u}, \mathbf{v} \rangle_{V' \times V} = (\mathcal{B}(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \\ \langle G(\boldsymbol{\sigma}, \mathbf{u}, \alpha), \mathbf{v} \rangle_{V' \times V} = (\mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \alpha), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \\ \langle \phi(\boldsymbol{\sigma}, \mathbf{u}, \alpha), \mathbf{v} \rangle_{V' \times V} = \langle S(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \alpha), \mathbf{v}) \rangle_{V' \times V} \\ \langle f, \mathbf{v} \rangle_{V' \times V} = \langle \mathbf{f}, \varepsilon(\mathbf{v}) \rangle_{V' \times V} \end{cases}$$

Obviously, $((.,.))_H$ and $(.,.)_H$ are equivalent inner products due to the assumption of mass density ρ. □

Proof.

Thus, we only need to verify that all the conditions (2)-(11) are satisfied. Clearly, (2)-(6) and (11) are met. Now we turn to check the remaining conditions. Since

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} \beta \left\| u_\nu \right\| \left(\mu \left\| \mathbf{v}_\tau - \mathbf{v}^* \right\| + v_\nu \right) da$$

and for any $v_1, v_2 \in V$

$$\left| \lambda \mathbf{v}_1 + (1-\lambda)\mathbf{v}_2 - \mathbf{v}^* \right| \leq \lambda \left| \mathbf{v}_1 - \mathbf{v}^* \right| + (1-\lambda) \left| \mathbf{v}_2 - \mathbf{v}^* \right|$$

we deduce that $j(u,.)$ is a proper convex functional □

Proof.

$$
\begin{aligned}
j(g, v_1) - j(g, v_2) &= \int_{\Gamma_3} \beta \|g_\nu\| \left(\mu \|v_{1,\tau} - v^*\| + v_{1,v}\right) da \\
&- \int_{\Gamma_3} \beta \|g_\nu\| \left(\mu \|v_{2,\tau} - v^*\| + v_{2,\nu}\right) da \\
&= \int_{\Gamma_3} \beta \|g_\nu\| \left(\mu \|v_{1,\tau} - v^*\| - \mu \|v_{2,\tau} - v^*\| + v_{1,\nu} - v_{2,\nu}\right) da \\
&\leq \int_{\Gamma_3} \beta \|g_\nu\| \left(\mu \|v_{1,\tau} - v_{2,\tau}\| + \|v_{1,\nu} - v_{2,\nu}\|\right) da \\
&\leq \|\beta\|_{L^\infty(\Gamma_3)} \|\mu\|_{L^\infty(\Gamma_3)} \|g_\nu\|_{L^2(\Gamma_3)} \|v_{1,\tau} - v_{2,\tau}\|_{L^2(\Gamma_3)} \\
&+ \|\beta\|_{L^\infty(\Gamma_3)} \|g_\nu\|_{L^2(\Gamma_3)} \|v_{1,\nu} - v_{2,\nu}\|_{L^2(\Gamma_3)} \\
&\leq \|\beta\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) \|g_\nu\|_{L^2(\Gamma_3)} \|v_{1,\nu} - v_{2,\nu}\|_{L^2(\Gamma_3)} \\
&\leq \|\beta\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) \|g\|_{L^2(\Gamma_3)} \|v_1 - v_2\|_{L^2(\Gamma_3)} .
\end{aligned}
\tag{62}
$$

Proof.

We know that there exists a constant $c_\gamma > 0$ such that

$$\|\mathbf{u}\|_{L^2(\Gamma_3)^d} \leq \|\mathbf{u}\|_{L^2(\Omega)^d} \leq c_\gamma \|\mathbf{u}\|_V, \quad \forall \mathbf{u} \in L^2\left(\Gamma_3\right)^d$$

Then the inequality (62) can be transformed as follows

$$j(\mathbf{g}, \mathbf{v}_1) - j(\mathbf{g}, \mathbf{v}_2) \leq c_\gamma^2 \|\beta\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) \|\mathbf{g}\|_{V)} \|\mathbf{v}_1 - \mathbf{v}_2\|_V$$

Thus the condition (9) holds. Similarly, we have

$$j(\mathbf{g}_1, \mathbf{v}_2) - j(\mathbf{g}_1, \mathbf{v}_1) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2)$$

$$\leq c_\gamma^2 \|\beta\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1\right) \|\mathbf{g}_1 - \mathbf{g}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V. \quad \square$$

and so the condition (8) is true. Therefore, we verify that all the conditions of Theorem5 are satisfied and so Problem P1 is uniquely solvable.

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THANK YOU FOR YOUR ATTENTION

Investigation of bifurcation de Hopf from the method hidden bifurcation in Chua system generated by Transformation

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Abstract

In this paper, an investigation of bifurcation de Hopf from the method of hidden bifurcation in the Chua system generated by Transformation is proposed. The verification of such a hidden bifurcation on two parameters corresponds to the procedure presented by Menacer, et al. (2016) for Chua multi scroll attractors. These hidden bifurcations (the number of the spiral) are governed by an additional homotopy parameter ε. Also, we discover a bifurcation de Hopf by controlling parameter k in the function, where we took the values of parameter k from 0.082 to 0.2 for search a bifurcation de Hopf.
Keywords: Chua system, Parallel transformation, hidden bifurcations

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SUR LES OPÉRATEURS DE TOEPLITZ TRONQUÉS NON-BORNS.

AMEUR YAGOUB [1]

Abstract. Ce travail s'intéresse aux opérateurs de Toeplitz tronqués, qui sont des compressions d'opérateurs de multiplication sur les espaces modèles. Nous donnons les conditions nécessaires et suffisantes pour qu'un opérateur de Toeplitz tronqué non-borné commute avec le Shift généralisé.

1. Introduction

on note par $\mathbb{T}$ le cercle unité, $dm := dm(\theta) = \frac{d\theta}{2\pi}$ la mesure de Lebesgue normalisée sur $\mathbb{T}$. $L^2(\mathbb{T}) := L^2(\mathbb{T}, dm)$, $L^\infty(\mathbb{T}) := L^\infty(\mathbb{T}, dm)$ les espaces de Lebesgue usuels sur $\mathbb{T}$.

1.1. **Espaces de Hardy et de De Branges-Rovnyak.** L'espace de Hardy H^2 est l'ensemble des fonctions $f \in L^2(\mathbb{T})$ tel le coefficient de Fourier négatives sont nulles.

$$H^2 = \{f \in L^2(\mathbb{T}), \widehat{f}(n) = 0, n < 0\}.$$

et

$$H^\infty = \{f \in L^\infty(\mathbb{T}), \widehat{f}(n) = 0, n < 0\},$$

où, $\widehat{f}(n) = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\frac{d\theta}{2\pi}, n \in \mathbb{Z}$, est le n-ème coefficient de Fourier de f.

Les oprateurs de Toeplitz sont les compressions des oprateurs de multiplication au l'espace de Hardy H^2, dfinie comme suit.

Definition 1.1. Soit $\varphi \in L^\infty(\mathbb{T})$, l'oprateur de Toeplitz avec le symbole φ est l'oprateur T_φ dfini par

$$\begin{array}{rcl} T_\varphi : H^2 & \longrightarrow & H^2 \\ f & \longmapsto & T_\varphi f = P(\varphi f), \end{array}$$

où P est la projection orthogonale de $L^2(\mathbb{T})$ sur H^2.

Soit $b \in H^\infty$, l'espace de de Brange-Rovnyak $\mathcal{H}(b)$ est l'image de H^2 par l'operator $(I - T_bT_{\overline{b}})^{1/2}$, c'est 'a dir

$$\mathcal{H}(b) = (I - T_bT_{\overline{b}})^{1/2}(H^2).$$

on a si $\|b\|_\infty < 1$ alors $\mathcal{H}(b) = H^2$.

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Key words and phrases. Espaces de Hardy, espaces modèle, fonction intérieure, opérateurs de Toeplitz et Toeplitz tronqués, opérateurs de Toeplitz tronqués non-bornés, Shift généralisé.

1.2. **Classes de Smirnov et Smirnov locale $\mathcal{N}^+, \mathcal{N}_u^+$.** La classe de Smirnov $\mathcal{N}^+$, est donn par

$$\mathcal{N}^+ = \{\varphi = \frac{b}{a}, a, b \in H^\infty, a \ extrieure, |a|^2 + |b|^2 = 1, p.p \ sur \ \mathbb{T}, a(0) > 0\}.$$

D'après [3], tout $\varphi \in \mathcal{N}_u^+$ a une reprsentation canonique unique $\varphi = \frac{b}{va}$, avec $a, b \in H^\infty$, a est une fonction extrieure, tel que $a(0) > 0, |a|^2 + |b|^2 = 1, pp \ sur \ \mathbb{T}$, v est intrieure, v, b et v, u sont relativement premiers,

$$\mathcal{N}_u^+ = \{\varphi = \frac{b}{va}, \frac{b}{a} \in \mathcal{N}^+, v \ extrieure, v, b \ et \ v, u \ sont \ relativement \ premiers\}.$$

1.3. **Espaces modèles.** L'opérateur de décalage à droite (ou shift) sur H^2 est défini par: $S[f](z) = zf(z), f \in H^2$, et son adjoint l'opérateur de décalage à gauche $S^* : H^2 \to H^2$ défini par: $S^*[f](z) = \frac{f(z)-f(0)}{z}, f \in H^2$. Un sous-espace $M \in H^2$, M est dit invariant par S s'il est fermé et tel que $SM \subset M$, et il dit que M non trivial lorsque $\{0\} \subsetneq M \subsetneq H^2$. Le théorème de Beurling donne une caractérisation complète des sous espaces invariant par le shift dans H^2, ils sont tous de la forme: $uH^2 = \{uh, h \in H^2\}$, où u est une fonction intérieure de H^2 (une fonction $u \in H^\infty$ est dit intérieure lorsque $|u^*| = 1$, sur $\mathbb{T}$)(voir [1]). Le fait que si uH^2 est un sous espace fermé invariant par S dans H^2 si et seulement si $(uH^2)^\perp$ est invariant par S^* implique que les sous espaces fermés non nul de H^2, invariants par S^* sont de la forme:

$$K_u := (uH^2)^\perp = H^2 \ominus uH^2 = \{f \in H^2, \langle f, ug \rangle, \forall g \in H^2\},$$

pour une certaine fonction intérieure $u \in H^2$. Le sous espace K_u est appelé espace modèle correspondant à la fonction intérieure u, qui est un sous-espace fermé de $L^2(\mathbb{T})$. On a si b est une fonction intrieure, alors $\mathcal{H}(b) = K_b$.
Chaque espace modèle est équipé d'un opérateur de conjugaison C défini par

$$C[f](\zeta) = u(\zeta)\overline{\zeta f(\zeta)} := \widetilde{f},$$

pour tout $f \in K_u$ et $\zeta \in \mathbb{T}$, et possède un noyau reproduisant, en-effet: pour toute $h \in H^2$, donc il existe dans K_u une unique fonction, notée k_λ^u telle que :

$$f(\lambda) = \langle f, k_\lambda^u \rangle, \qquad f \in K_u$$

où

$$k_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{(1 - \overline{\lambda}z)}, \quad \lambda \in \mathbb{D}, z \in \mathbb{T}, \tag{1.1}$$

donc la fonction k_λ^u est appelée le noyau reproduisant pour K_u.

1.4. **Opérateurs de Toeplitz tronqués borns.** Les opérateurs de Toeplitz tronqués sont des compressions des opérateurs de multiplication sur l'espace modèle K_u. Les opérateurs de Toeplitz tronqués ont été formellement introduits par Sarason dans [2]. L'étude des opérateurs de Toeplitz tronqués est un nouveau domaine de recherche dans la théorie des opérateurs dont beaucoup de questions de base concernant restent ouvertes. Dans toute la suite, on fixe u

comme étant une fonction intérieure. les compressions de S et S^* sur K_u^2 sont notées respectivement par S_u et S_u^* c'est-à-dire.

$$S_u = S|_{K_u}, \quad S_u^* = S^*|_{K_u}.$$

Ces deux opérateurs jouent des rles cruciaux dans la théorie des opérateurs de Toeplitz tronqués. Comme chaque noyau reproduisant de (1.1) est analytique borné et $span\{k_\lambda^u, \lambda \in \mathbb{D}\}$ (le sous-espace vectoriel fermé engendré par k_λ^u) est dense dans K_u, il s'ensuit que $K_u \cap H^\infty := K_u^\infty$ est dense dans K_u.

Definition 1.2. L'opérateur de Toeplitz tronqué de symbole $\varphi \in L^2(\mathbb{T})$ sur K_u^∞ est défini par:

$$\begin{array}{rcl} A_\varphi^u : K_u^\infty & \longrightarrow & K_u^\infty \\ f & \longmapsto & A_\varphi^u(f) = P_u(\varphi f), \end{array}$$

avec P_u est la projection orthogonale de $L^2(\mathbb{T})$ sur K_u.

Les oprateurs de shifts gnraliss ont t dfinis par Sarason ([2]), ils sont la somme deux oprateurs de Toeplitz tronqus.

Definition 1.3. Soit $\alpha \in \overline{\mathbb{D}}$, l'oprateur de shift gnralis est dfini par

$$S_u^\alpha := S_u + \frac{\alpha}{1 - \overline{u(0)}\alpha} k_0^u \otimes \widetilde{k_0^u}. \tag{1.2}$$

Si $\alpha = 0$, alors $S_u^0 = S_u$, et si $\alpha = \infty$, alors $S_u^\alpha = S^*$.

1.5. **Opérateurs de Toeplitz et Toeplitz tronqués non-bornés.** [3]. Soit φ est une fonction non nulle dans $\mathcal{N}^+$, avec représentation canonique $\varphi = \frac{b}{a}$. L'opérateur T_φ, l'opérateur Toeplitz avec symbole φ, est par définition l'opérateur de multiplication par φ sur le domaine

$$D(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\} = aH^2.$$

L'opérateur T_φ est fermé et densément défini, et son adjoint fermé et densément défini T_φ^*. Le domaine $D(T_\varphi^*) = \mathcal{H}(b)$, et le graphique

$$\mathcal{G}(T_\varphi^*) = \{f \oplus g \in H^2 \oplus H^2 : T_{\overline{b}} f = T_{\overline{a}} g\}.$$

L'opérateur $T_{\overline{\varphi}}$, l'opérateur de Toeplitz avec symbole $\overline{\varphi}$, est défini comme T_φ^*.

L'opérateur $T_{\overline{\varphi}}$ induit un opérateur sur K_u, désigné par $A_{\overline{\varphi}}^u$, et défini par

$$A_{\overline{\varphi}}^u = T_{\overline{\varphi}} / D(T_{\overline{\varphi}}) \cap K_u,$$

avec domaine

$$D(A_{\overline{\varphi}}^u) = \mathcal{H}(b) \cap K_u.$$

Dans [3], Sarason a été défini l'opérateur A_φ^u, on fonction de $\varphi \in \mathcal{N}_u^+$, il utilise le calcul fonctionnel pour définir ces opérateurs: pour $\varphi \in \mathcal{N}_u^+$, avec la représentation canonique unique $\varphi = \frac{b}{va}$, on a

$$A_{\overline{\varphi}}^u = ((va)^*(S_u^*))^{-1} b^*(S_u^*),$$

o 'u, pour une fonction ψ holomorphe dans $\mathbb{D}$, $\psi^*(z) = \overline{\psi(\overline{z})}$. Alors $A_{\overline{\varphi}}^u$ est opérateur fermé densément défini dans K_u. L'espace K_u admet un opérateur de conjugaison naturelle C, donc A^u_φ est la transformée de $A^u_{\overline{\varphi}}$ avec l'opérateur C, c'est- 'a-dire ,

$$\begin{aligned} D(A^u_\varphi) &= CD(A^u_{\overline{\varphi}}) = \{f : Cf \in \mathcal{H}(b) \cap K_u\} \\ A^u_\varphi Cf &= CA_{\overline{\varphi}} f, \forall f \in D(A^u_{\overline{\varphi}}). \end{aligned}$$

Dans [3], Sarason prolonge les résultats de Suárez dans [5],dans le théor 'eme suivant

Theorem 1.4. *Un opérateur fermé A densément défini dans K_u commute avec S_u si et seulement si $A = \varphi(S_u)$ où $\varphi \in \mathcal{N}^+_u$.*

Quelques propriétés intéressantes dans les articles de Sarason [3, 4], sont regroupées, pour $\varphi \in \mathcal{N}^+_u$:

(*i*) Les opérateurs A^u_φ et $A^u_{\overline{\varphi}}$ sont adjoints l'un de l'autre.

(*ii*)
$$\mathcal{G}(A^u_\varphi) = \{f \oplus g \in K_u \oplus K_u : A^u_b f = A^u_{va} g\},$$
et
$$\mathcal{G}(A^u_{\overline{\varphi}}) = \{f \oplus g \in K_u \oplus K_u : A^u_{\overline{b}} f = A^u_{\overline{va}} g\}.$$

(*iii*) Soit $\psi \in H^\infty$, alors
$$A^u_{\overline{\psi}} A^u_{\overline{\varphi}} f = A^u_{\overline{\varphi}} A^u_{\overline{\psi}} f = A^u_{\overline{\varphi \psi}} f$$
pour f dans $D(A^u_{\overline{\varphi}})$.

(*iv*) $A^u_{\overline{\varphi}} f = A^u_{1/\overline{v}} A^u_{\overline{b/a}} f$ pour f dans $D(A^u_{\overline{\varphi}})$.

(*v*) Soient φ_1 et φ_2 deux fonctions non nulles dans $\mathcal{N}^+_u$. Alors $A^u_{\overline{\varphi_1}} = A^u_{\overline{\varphi_2}}$ si et seulement si u divise $\varphi_1 - \varphi_2$.

Objectif. Dans le présent travail, nous donnons une condition nécessaire et suffisante pour qu'un opérateur non-borné commute avec le Shift généralisé S^α_u.

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Quasi-variational inequalities of hyberbolic type for a dynamic viscoelastic contact frictional problem

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Abstract

The aim of this work is to prove the existence of solution for a quasi-variational inequalities of hyberbolic type by using a time-discretization technique and monotone operators theory. We illustrate the abstract results by an application to dynamic contact frictional problem for viscoelastic materials.

Keywords: Quasivariational inequalities, hyperbolic, time-discretization technique, frictional.

1 Introduction

In this paper we establish the existence of solution to hyperbolic quasivariational inequalities with the viscosity term arising in dynamic viscoelastic frictional contact problem, where the friction contact is modeled by the Coulomb law of dry friction,

Here, V is a Banach space of admissible displacements, and we introduce A and B are operators related to the viscoelastic constitutive law, φ is a convex functional related to contact boundary conditions, and . The function f represents the given body forces and surface traction, and u_0, u_1 represents the initial displacement and velocity, respectively. Moreover, the derivative of displacement u' is with respect to the time variable t, the interval of interest is $[0, T]$.

2 Existence of solution

In this section , we consider an evolution of triple spaces $V \subset H \subset V^*$, where V is a strictly convex, reflexive and separable Banach space, H is a separable Hilbert space. For $0 < T < +\infty$, we consider the standard Bochner-Lebesgue function spaces $\mathcal{V} = L^2(0, T; V)$ and $\mathcal{W} = \{v \in \mathcal{V} | v' \in \mathcal{V}^*\}$, where $v' = \partial v/\partial t$ is the time derivative in the sense of vector-valued distributions. By the reflexivity of V we have both $\mathcal{V}$ and its dual $\mathcal{V}^* = L^2(0, T; V^*)$ are reflexive Banach spaces.

Let $A, B : V \to V^*$ are a linear operators, a functional $\varphi : V \times V \to \mathbb{R} \cup \{+\infty\}$ and $f \in \mathcal{V}^*, u_0, v_0 \in V$, we consider the hyperbolic quasivariational inequality of finding an element $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ with some hypotheses H(A),H(B), H(φ), H(f) and H(aux),

We make the following hypotheses.

H(A): $A : [0, T] \times V \to 2^{V^*}$ is multivalued operator such that

(a) $A(., v) : [0, T] \times V \to 2^{V^*}$ is measurable for all $v \in V$;

(b) $A(t, .)$ is linear and maximal monotone for a.e. $t \in [0, T]$; there exist $a_1 \in L^q(0, T)$ and a constant $c_1 > 0$ such that for all $v \in V$

$$\|\psi(t)\|_{V^*} \leq a_1(t) + c_1\|\psi(t)\|_V^{p-1}, \ \ \forall \ \ \psi(t) \in A(t, v) \ \text{ and } \ \text{a.e}\ t \in [0, T]$$

(c) there exist $a_2 \in L^1_+(0, T)$ and $c_2 > 0$ such that for all$v \in V$

$$\langle \psi(t), v\rangle \geq c_2\|v\|_V^p - a_2(t), \ \ \forall \ \ \psi(t) \in A(t, v) \ \text{ and } a.e\ t \in [0, T]$$

H(B) $B : V \to V^*$ is linear, bounded, symmetric and monotone, i.e.,

(a) $B \in \mathcal{L}(V, V^*)$ and $\forall \ v \in V, \ \|B(v)\|_{V^*} \leq c_3\|v\|_V$ with $c_3 > 0$;

(b) $\langle B(u), v\rangle = \langle B(v), u\rangle \geq 0, \forall u, v \in V$

H(φ) $\varphi : M \times M \to \mathbb{R}$ is such that

(a) $\varphi(., u)$ is measurable for all $u \in M$ and $\varphi(u, .)$ is proper, convex and lower semicontinuous for a.e. $t \in [0, T]$

(b) there exist a function $a_3 \in L^q(0, T)$ and $c_4 > 0$ such that

$$\|\eta\|_{M^*} \leq a_3(t) + c_4\|v\|_M^{p-1}, \ \forall v \in M, \ \eta \in \partial\varphi(t, v) \ \ a.e.t \in [0, T]$$

(c) the mapping $\partial\varphi(., .)$ is upper semicontinuous endowed with the weak topology from $M \times M$ to M^*.

H(γ) the operator $\gamma : V \to M$ is linear and compact with its adjoint operator γ^*.

H(f) $f \in L^q(0, T; V^*)$ and $u_0 \in V$.

We have the following theorem of existence.

Theorem 2.1. *Assume that assumptions H(A),H(B),H(φ), H(f) and H(γ) holds. Then the hyberbolic quasivariational inequality has a solution.*

The proof of Theorem 2.1 is based on three basic steps.

1. We define time discrete family problems corresponding to the quasi-variational inequalities.

2. We reformulate the time discrete family problems as inclusion which solved by surjectivity theorem.

3. We prove a convergence result. Hence, we deduce that the hyperbolic quasivariational inequality has a solution.

3 A Dynamic contact frictional problem for viscoelastic materials

In this section, we consider a dynamic contact frictional problem for viscoelastic materials and we prove existence of weak solution by using the abstract result in Section 2. The friction condition is described with the evolutionary version of Coulomb law of dry friction, More details can be found in [2].

4 Conclusion

As conclusion, it is evident that this study has shown that the hyperbolic quasivariational inequality has a solution. Further study of the issue would be of interest when the viscosity term is vanished in order to obtain an existence result for an elastodynamic Signorini problem with Coulomb friction law.

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[On the existence of positive solutions of a degenerate paraolic system applied in ecology]

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Abstract: The aim of this paper is to show the existence, uniqueness. Also investigated is the existence of positive maximal and minimal solutions of the corresponding quasilinear elliptic system. The elliptic operators in both systems are allowed to be degenerate in the sense that the density-dependent diffusion coefficients $D_i(u_i)$ may have the property $D_i(0) = 0$ for some or all $i = 1, ..., N$, and the boundary condition is $u_i = 0$. Using the method of upper and lower solutions, we show that a unique global classical time-dependent solution exists and converges to the maximal solution for one class of initial functions and it converges to the minimal solution for another class of initial functions; and if the maximal and minimal solutions coincide then the steady-state solution is unique and the time-dependent solution converges to the unique solution.
Keywords : Parabolic system, degenerate reaction diffusion system, method of upper and lower solutions.
2010 Mathematics Subject Classification : 35K50, 35J55, 35K57.

1 Introduction

Degenerate quasilinear parabolic and elliptic equations have received extensive attentions during the past several decades and many topics in the mathematical analysis are well developed and applied to various fields of applied sciences.
In this paper, we consider a coupled system of arbitrary number of quasilinear paraolic equations in a bounded domain with Dirichlet boundary condition where the domain is assumed to have the outside sphere property without the usual smoothness condition. The

system of equations under consideration is given by

$$\begin{cases} \frac{\partial u_i}{\partial t} - \operatorname{div}\left(D_i\left(u_i\right)\nabla u_i\right) + b_i.\left(D_i\left(u_i\right)\nabla u_i\right) = f_i\left(t,x,u\right) \quad , \quad x \in Q_T \\ u_i\left(t,x\right) = g_i\left(t,x\right) \quad , \quad \left(t,x\right) \in S_T \\ u_i\left(0,x\right) = h_i\left(x\right) \quad , \quad x \in \Omega, \quad i = 1,...,N \end{cases} \tag{1}$$

Now, we assume the following assumptions :
where $u = (u_1, ..., u_N)$, Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$, and for each $i = 1, ..., N$, $b_i = b_i\left(t,x\right)$ are independent of t, $D_i\left(u_i\right)$, $f_i\left(t,x,u\right)$ and $g_i\left(t,x,u\right)$ are prescribed functions satisfying the following hypotheses :

(H_1) $b_i^{\ell}\left(t,x\right)$ for $\ell = 1, ..., n$ and $f_i\left(t,x,.\right)$ are in $C^{\frac{\alpha}{2},\alpha}\left(\bar{\Omega}\right)$, $g_i\left(t,x\right) \in C^{\frac{\alpha}{2},\alpha}\left(\partial\Omega\right)$.

(H_2) $D_i\left(u_i\right) \in C^2\left([0, M_i]\right), D_i\left(u_i\right) > 0$ for $u_i \in [0,\ M_i]$, and $D_i\left(0\right) = 0$, where $M_i = |\tilde{u}|_{C(\bar{\Omega})}$.

(H_3) $f_i\left(t,x,.\right) \in C^{\alpha}\left(\bar{\Omega}\right)$, $f_i\left(.,.,u\right) \in C^1\left(S^*\right)$, and

$$\frac{\partial f_i}{\partial u_j} \geq 0\ ,\ for\ j \neq i\ ,\ u \in S^*$$

(H_4) $g_i\left(t,x\right) \geq 0$ on S_T, $h_i\left(x\right) > 0$ in Ω.

(H_5) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball K outside of Ω with radius $r \geq \delta_0$ such that $K \cap \bar{\Omega} = \{x_0\}$.

In the above hypothesis, S^* is the sector between a pair of upper and lower solutions given by

$$S^* = \left\{u \in C\left(\bar{\Omega}\right) \mid \hat{u} \leq u \leq \tilde{u}\right\}$$

below. It is allowed that $D_i\left(0\right) = 0$ for some i and $D_i\left(0\right) > 0$ for a different i. In particular, if $D_i\left(u\right)$ is a positive constant for all i then system (1) becomes the standard coupled system of semilinear parabolic equations. Our approach to the existence problem is again by the method of upper and lower solutions which are defined as follows:

Definition 1 *A pair of functions $\tilde{u}_s \equiv (\tilde{u}_1, ..., \tilde{u}_N)$, $\hat{u}_s \equiv (\hat{u}_1, ..., \hat{u}_N)$ in $C^2\left(\Omega\right) \cap C\left(\bar{\Omega}\right)$ are called ordered upper and lower solutions of* (1) *if $\tilde{u}_s$, $\hat{u}_s$, and if*

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla.\left(D_i\left(u_i\right)\nabla u_i\right) + b_i.\left(D_i\left(u_i\right)\nabla u_i\right) \leq f\left(t,x,u\right) \quad , \quad \left(t,x\right) \in Q_T \\ u_i\left(t,x\right) \leq g_i\left(t,x\right) \quad , \quad \left(t,x\right) \in S_T \\ u_i\left(0,x\right) = h_i\left(x\right) \quad , \quad x \in \Omega, \quad i = 1,...,N \end{cases} \tag{1.2}$$

and $\tilde{u}_i$ satisfies (2) *with inequalities reversed.*

2 The main result

The main result of this work is the following theorem :

Theorem 1 *Let $\tilde{u}_s$,$\hat{u}_s$ be ordered positive upper and lower solutions of* (1)*, and let hypotheses $(H_1) - (H_4)$ hold. Then problem* (1) *has a minimal solution $\underline{u}_s$ and a maximal solution $\bar{u}_s$ such that $\hat{u}_s \leq \underline{u}_s \leq \bar{u}_s \leq \tilde{u}_s$. If $u_s = \bar{u}_s\ (\equiv u_s^*)$ then u_s^* is the unique positive solution in S^*.*

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Boubaker Operational Matrix Method for fractional Emden-Fowler problem

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Abstract

The objective of this paper is to find numerically the solution of singular Emden-Fowler equations of fractional order . For the approximation of the solutions we have used Boubaker polynomials . The operational matrix of the Caputo fractional derivative tool converts these problems to systems of algebraic equations whose solutions are simple and easy to compute. Numerical applications have proven the effectiveness of these methods to find approximate and precise solutions.

Keywords:
Boubaker Polynomials ; Operational matrix of fractional derivatives ; Collocation method; Fractional Emden-Fowler Type Equations .

Mathematics Subject Classification 2010: 65L05 34A08, 65L60, 26A33

1 Introduction

In mathematical physics and nonlinear mechanics there exists sufficiently large number of particular basic singular Fractional differential equations for which an exact analytic solution in terms of known functions did not exist [9,10,14,17,22]. One of these equations describing many phenomena in mathematical physics and astrophysics such as, the thermal behavior of a spherical, cloud of gas, isothermal gas sphere and theory of stellar structure, theory of thermionic currents etc . Is called the singular Emden-Fowler equations of fractional order formulated as: [12,18,20,21]

$$D^{2\alpha}u(x) + \frac{\lambda}{x^{\alpha}} D^{\alpha}u(x) + s(x)g(u(x)) = h(x), \quad x \in (0,1), \quad \lambda > 0, \quad \frac{1}{2} < \alpha \leq 1 \tag{1.1}$$

subject to the conditions:

$$u\left(0\right) = a, \quad D^{\alpha}u\left(0\right) = b$$

where a and b are constants. When $\alpha = 1, \lambda = 2$, and $h(x) = 1$, Eq.(1.1) becomes the Lane-Emden type equation. D^{α} denote the Caputo fractional derivatives. It is generally defined as follows :

$$D^{\alpha}u\left(x\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_0^x \frac{u^{(n)}\left(t\right)}{\left(x-t\right)^{\alpha-n+1}} dt, \quad n-1 < \alpha < n, , \quad n \in \mathbb{N}, , \quad \alpha > 0 \tag{1.2}$$

For the Caputo derivative we have $D^{\alpha}C = 0$ where C is a constant, and

$$D^{\alpha}x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}\cup\{0\} \text{ and } \beta < \lceil\alpha\rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha} & \text{for } \beta \in \mathbb{N}\cup\{0\} \text{ and } \beta \geq \lceil\alpha\rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lceil\alpha\rceil \end{cases} \tag{1.3}$$

where, $\lceil\alpha\rceil$ denotes the largest integer less than or equal to α and $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

The problem (1.1) is studied by using the Residual Power Series Method [20], Homotopy analysis method (HAM) [12], Reproducing kernel Hilbert space method [21], The fractional differential transformation (FDT) [18], Polynomial Least Squares Method [6], Shifted Legendre Operational Matrix [**?**], Chebyshev wavelets [13], Orthonormal Bernoullis polynomials [23], Orthonormal Bernstein polynomials [1]. For the solution of the classic Emden-Fowler equations (Case $\alpha = 1$), there are many Studies of analytical as well as numerical methods is provided in monographs [2, 7, 8, 24, 26–28].

The purpose of this paper is to use Boubaker operational matrix of fractional for Solving a singular initial value problems of fractional Emden-Fowler type equations (1.1). To the best of our knowledge this is the first time that the Boubaker operational matrices are used to obtain solutions singular Emden-Fowler equations of fractional order. First we present a new theorem which can reduce the fractional Emden-Fowler problem toa system of algebraic equations. The Boubaker polynomials were established for the first time by by Boubaker (2007), to solve heat equation inside a physical model. The first monomial definition of the Boubaker polynomials on interval $x \in [0, 1]$, was introduced by [3–5, 11, 16, 19]:

$$\mathbf{B}_0(x) = 1, \quad \mathbf{B}_n(x) = \sum_{p=0}^{\xi(n)}[\frac{(n-4p)}{(n-p)}C_{n-p}^{p}](-1)^p x^{n-2p}, \quad n \geq 1. \tag{1.4}$$

where $\xi(n) = \lfloor\frac{i}{2}\rfloor = \frac{2n+((-1)^n-1)}{4}$ and $C_{n-r}^{r} = \frac{(n-p)!}{r!(n-2p)!}$. The symbol $\lfloor.\rfloor$ denotes the floor function. The Boubaker polynomials could be calculated by following recursive formula:

$$\mathbf{B}_m(x) = x\mathbf{B}_{m-1}(x) - \mathbf{B}_{m-2}(x), \quad m \geq 2. \tag{1.5}$$

We will construct operational matrix of Caputo fractional derivatives $\mathbf{D}^{(\alpha)}$ for the Boubaker polynomials are given

$$D^{\alpha}\mathbf{B}(x) \simeq \mathbf{D}^{(\alpha)}\mathbf{B}(x) \tag{1.6}$$

where $\mathbf{B}(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T$ be Boubaker vector and the matrice $\mathbf{D}^{(\alpha)}$ are of order $(N+1) \times (N+1)$. In order to show the high importance of Boubaker operational matrix of fractional derivative, we apply it to solve equation (1.1).

The paper is organized as follows. In Section (**??**), express Boubaker polynomials in terms of Taylor basis, and Function approximation. In Section (**??**) The operational matrix of Caputo fractional derivatives is constructed. In Section(**??**), we use Boubaker polynomials method for Solving Fractional Emden-Fowler Type Equations. Section(**??**) illustrates some numerical examples to show the accuracy of this method. Finally, Section (2) concludes the paper

2 Conclusions

In this work, we get operational matrices of the fractional derivative by Boubaker polynomials. Then by using these matrices, we reduced the singular Fractional Emden-Fowler Type Equations to a system of algebraic equations that can be solved easily. A Numerical example is included to demonstrate the validity and applicability of this method, and the results reveal that the method is very effective, straightforward, simple, and it can be applied by developing for the other related fractional problem, such as such as partial fractional differential and integro-differential equations, This is area of our future work

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Feedback boundary stabilization of the Schrödinger equation with interior delay

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Abstract

In [1] Ammari et al established, under Lions geometric condition, an exponential stability result for the wave equation with an interior delay term and a Neumann boundary feedback. Boundary stabilization problems for the undelayed Schrödinger equation were considered in [2] and [3]. In [4], stability problems for the Schrödinger equation with a delay term in the boundary or internal feedbaks were investigated. Our aim in this paper is to study the boundary stabilization problem for the Schrödinger equation with an interior time delay. Under suitable assumptions, we prove exponential stability of the solution. This result is obtained by using multiplier techniques and by introducing a suitable lyapunov functional.

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Key Words and Phrases: *Schrödinger equation, interior delay, boundary stabilization.*

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Behavior of the solutions of some systems of non-integer differential equations : comparison and principle of the maximum

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Absrtact

This work is devoted to the generalization of certain theorems of comparisons known in the theory of ordinary differential equations,to systems of differential equations of fractional order.

Keywords : Fractional integral, Riemann-Liouville derivative, Caputo derivative, fractional differential system, differential inequality, integral inequality, comparison theorems.

The objective of this work

The aim of this work is to study the comparison of solutions, in cases where the differential problem is in the form of a system of fractional differential equations. So, we are interested in the problems where the unknowns are vector functions : $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. Thus, we consider the systems of inequalities of the following form.

$$
\begin{aligned}
{}^{c}D_{0^+}^{\beta_i} u_i(t) &\leq f_i\left(t, u(t)\right) \\
{}^{c}D_{0^+}^{\alpha_i} v_i(t) &\geq f_i\left(t, v(t)\right), \ 0 \leq t \leq T,
\end{aligned}
$$

where $i = \overline{1, n}$: $0 < \alpha_i \leq 1$ and $0 \leq \beta_i < 1$, and $f_i \in C\left([0, T] \times \mathbb{R}^n; \mathbb{R}\right)$
We will start with a first analysis on systems of order two (in $\mathbb{R}^2$).

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Existence result for fractional boundary value problem with fractional integral boundary conditions on the unbounded interval

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Abstract

In this paper, we are concerned with the existence of solutions to fractional differential equation subject to Riemann-Liouville fractional integral boundary conditions. By means of a recent fixed point theorem, sufficient conditions are obtained that guarantee the existence of at least one solution. Two examples of applications illustrate the applicability of the theoretical result.

Keywords: Boundary value problem; fractional differential equation; infinite interval; fixed point theorem.

1 Introduction

In this paper, we will consider the boundary value problem (bvp for short)

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t), D_{0^+}^{\alpha-1}u(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) = 0, \qquad \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \beta I_{0^+}^{\alpha-1}u(\eta), \end{cases} \tag{1.1}$$

where $1 < \alpha \leqslant 2$, $\eta > 0$ and $\beta > 0$ satisfies $0 < \beta\eta^{2\alpha-2} < \Gamma(2\alpha-1)$. $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional derivative and $I^{\alpha-1}_{0^+}$ is the standard Riemann-Liouville fractional integral.

Fractional equations are natural generalization of the classical integer-order differential equations. They turn out to be very adequate for modeling dynamics of many processes involving complex systems that can found in science, engineering, aerodynamics, etc. For instance, integral boundary conditions appear in blood flow models as well as in chemical engineering. Fractional integral conditions may provide good information rather than classical local boundary conditions.

In the last years, many research papers have been concerned with fractional differential equations and great progress has been made in the study of bvps involving differential equations; see, e.g., [2]- [7], [9]- [10], [12]- [17] and references therein. For instance, A. Guezane-Lakoud, R. Khaldi [7] have studied the following boundary value problem with fractional integral boundary conditions in bounded interval

$$\begin{cases} {}^cD^{q}_{0^+}x(t) + f(t, x(t), {}^cD^{p}_{0^+}x(t)) = 0, \quad 0 < t < 1,\ 1 < q \leqslant 2,\ 0 < p < 1 \\ x(0) = 0, \quad x'(1) = \alpha I^{p}_{0^+}x(1), \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative.

In [12], C. Shen, H. Zhou, and L. Yang have established existence of positive solutions for the bvp

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t), D^{\alpha-1}_{0^+}u(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D^{\alpha-1}_{0^+}u(+\infty) = \sum_{i=1}^{m-2}\beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha \leqslant 3$. Using the Schauder fixed point theorem, they have showed the existence of one solution under suitable growth conditions imposed on the nonlinear term. X. Su and S. Zhang [13] discussed the existence of unbounded solutions and used

Schauder's fixed point theorem to prove existence of solutions for the bvp

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t), D_{0^+}^{\alpha-1}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D_{0^+}^{\alpha-1}u(\infty) = u_\infty, & u_\infty \in \mathbb{R}, \end{cases}$$

where $1 < \alpha \leqslant 2$.

C. Yu, J. Wang, and Y. Guo [16] have considered the solvability for the following integral boundary value problem of fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t), D_{0^+}^{\alpha-1}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad D_{0^+}^{\alpha-1}u(\infty) = \int_\eta^{+\infty} g(t)u(t)dt, \end{cases}$$

where $1 < \alpha \leqslant 2$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\eta \geqslant 0$, $g(t) \in L^1[0, +\infty)$ and $\int_\eta^{+\infty} g(t)u(t)dt < \Gamma(\alpha)$.

The work presented in this paper is a continuation of previous works and is concerned with a boundary value problem of fractional order set on the half-axis. It is mainly motivated by papers [7], [12], [13], [16]. To overcome compactness difficulty of fixed point point operator, a special Banach space is introduced. Our results allow the integral condition to depend on the fractional integral $I_{0^+}^{\alpha-1}u$ which leads to additional difficulties. The plan of the paper is as follows. First we present in Section 2 some definitions and lemmas which are crucial to our discussion. Related lemmas necessary to the fixed point formulation are given in Section 3. Section 4 is devoted to the main existence theorem of solutions. We end the paper by presenting two examples of application in order to illustrate the theoretical result.

2 Preliminaries

We first collect some definitions and lemmas on the fractional calculus (see [8], [11] for more details).

One of the basic tools of the fractional calculus is the Gamma function which extends the factorial to positive real numbers (and even to complex numbers with positive real parts).

Definition 2.1. *For $\alpha > 0$, the Euler gamma function is defined by*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Definition 2.2. *Let $p > 0,\ q > 0$, the Euler beta function is defined by*

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt.$$

Proposition 2.1. *Let $\alpha > 0,\ p > 0,\ q > 0$ and n a positive integer. Then*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

hence

$$\Gamma(\alpha+n) = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\Gamma(\alpha).$$

Special cases:

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}.$$

Definition 2.3. *The fractional integral of order $\alpha > 0$ for function h is defined as*

$$I_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided the right side is point-wise defined on $(0, +\infty)$.

Definition 2.4. *For a function h given on the interval $[0, +\infty)$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of h is defined by*

$$D_{0+}^{\alpha} h(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$.

Lemma 2.1. *([8]) Let $\alpha > 0$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\alpha] + 1$.

Proposition 2.2. *In this work, we need the following composition relations:*

(a) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} h(t) = h(t), \quad \alpha > 0, \quad h(t) \in L^1[0, +\infty)$.

(b) $D_{0^+}^{\alpha} I_{0^+}^{\gamma} h(t) = I_{0^+}^{\gamma-\alpha} h(t), \quad \gamma > \alpha > 0, \quad h(t) \in L^1[0, +\infty)$.

(c) $I_{0^+}^{\alpha} I_{0^+}^{\gamma} h(t) = I_{0^+}^{\alpha+\gamma} h(t), \quad \alpha > 0, \quad \gamma > 0 \quad h(t) \in L^1[0, +\infty)$.

(d) $D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$, for $\lambda > -1$, giving in particular $D_{0^+}^{\alpha} t^{\alpha-m} = 0$, $m = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

(e) $I_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)} t^{\alpha+\lambda}, \quad \alpha > 0, \quad \lambda > -1$.

(f) (Properties 1.3, of the Gamma function, in [11]) if $1 \leqslant \alpha \leqslant 2$ then $\frac{1}{\Gamma(\alpha)} \geqslant 1$.

The following result is needed to prove our main existence result. This is a nonlinear alternative for Krasnosel'skii' s fixed point theorem [1].

Theorem 2.1. *([1]) Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$, $F(\overline{U})$ bounded and $F : \overline{U} \to C$ is given by $F = F_1 + F_2$, where $F_1 : \overline{U} \to E$ is continuous and completely continuous and $F_2 : \overline{U} \to E$ is a contraction, i.e., there exists a constant $0 < l < 1$, such that $\|F_2(x) - F_2(y)\| \leqslant l\|x - y\|$, for all $x, y \in \overline{U}$. Then either,*

$(A1)$ F has a fixed point in $\overline{U}$, or

$(A2)$ There is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

3 Related Lemmas

Define the spaces

$$X = \left\{ u \in C([0, +\infty), \mathbb{R}) : \quad \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha}} < +\infty \right\}$$

with the norm

$$\|u\|_X = \sup_{t\geq 0} \frac{|u(t)|}{1+t^\alpha}$$

and

$$Y = \left\{ u \in X, \quad D_{0^+}^{\alpha-1}u \text{ exists}, \quad D_{0^+}^{\alpha-1}u \in C([0,\,+\infty),\mathbb{R}), \quad \sup_{t\geq 0} |D_{0^+}^{\alpha-1}u(t)| < +\infty \right\}$$

with the norm

$$\|u\|_Y = \max\left\{ \sup_{t\geq 0} \frac{|u(t)|}{1+t^\alpha}, \quad \sup_{t\geq 0} |D_{0^+}^{\alpha-1}u(t)| \right\}.$$

Lemma 3.1. *$(X,\ \|.\|_X)$ and $(Y,\ \|.\|_Y)$ are Banach spaces.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the space $(X,\ \|.\|_X)$. Then

$$\forall\,\varepsilon > 0, \exists\, N > 0 \text{ such that } \|u_n - u_m\|_X < \varepsilon \text{ for any } n, m > N,$$

i.e.

$$\forall\,\varepsilon > 0, \exists\, N > 0 \text{ such that } \left| \frac{u_n(t)}{1+t^\alpha} - \frac{u_m(t)}{1+t^\alpha} \right| < \varepsilon \text{ for any } t \in [0, +\infty) \text{ and } n, m > N.$$

Thus $\left\{\frac{u_n(t)}{1+t^\alpha}\right\}_{n\in\mathbb{N}}$ for $t \in [0,+\infty)$, is a Cauchy sequence in $\mathbb{R}$, too.
So, there exists a $l(t) \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} \left|\frac{u_n(t)}{1+t^\alpha} - l(t)\right| = 0,\ t \in [0,+\infty)$, this implies $\lim\limits_{n\to+\infty} \|u_n - u\|_X = 0$, where $u(t) = l(t)(1+t^\alpha)$. So $(X,\ \|.\|_X)$ is a Banach space.
In a similar way, we prove that $(Y, \|.\|_Y)$ is a Banach space.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(Y, \|.\|_Y)$. Then $\{u_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in the space $(X, \|.\|_X)$ and thus $\lim\limits_{n\to+\infty} \frac{u_n(t)}{1+t^\alpha} = \frac{u(t)}{1+t^\alpha}$ for some $u \in X$, i.e.

$$\forall\,\varepsilon > 0, \exists\, N > 0 \text{ such that } \left| \frac{u_n(t)}{1+t^\alpha} - \frac{u(t)}{1+t^\alpha} \right| < \varepsilon \text{ for any } t \in [0, +\infty) \text{ and } n > N.$$

Then there exists a positive constant $M > 0$ such that $\frac{|u_n(t)|}{1+t^\alpha} \leqslant M, \quad n \in \mathbb{N}$.
We claim that $\{D_{0^+}^{\alpha-1}u_n(t)\}_{n\in\mathbb{N}}$ converges. In fact, if $\alpha = 2$ we have

$$\lim_{n\to+\infty} u_n'(t) = \frac{d}{dt}\left(\lim_{n\to+\infty} u_n(t) \right) = u'(t),$$

and in the case $1 < \alpha < 2$, we get

$$\lim_{n\to+\infty} D_{0^+}^{\alpha-1}u_n(t) = \frac{1}{\Gamma(2-\alpha)} \lim_{n\to+\infty} \frac{d}{dt}\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)\frac{u_n(s)}{1+s^\alpha}ds.$$

Moreover, for any $t \in [0, +\infty)$ and $1 < \alpha < 2$, we get

$$\left|\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)\frac{u_n(s)}{1+s^\alpha}ds\right| \leqslant M\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)ds.$$

Since

$$\begin{aligned}\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)ds &= \int_0^1 t^{1-\alpha}(1-\tau)^{1-\alpha}(1+t^\alpha\tau^\alpha)td\tau \\ &= t^{2-\alpha}\int_0^1 (1-\tau)^{1-\alpha}d\tau + t^2\int_0^1 \tau^\alpha(1-\tau)^{1-\alpha}d\tau \\ &= \frac{1}{2-\alpha}t^{2-\alpha} + B(\alpha+1, 2-\alpha)t^2,\end{aligned}$$

then

$$\left|\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)\frac{u_n(s)}{1+s^\alpha}ds\right| \leqslant \frac{M}{2-\alpha}t^{2-\alpha} + B(\alpha+1, 2-\alpha)Mt^2.$$

According to the Lebesgue's dominated convergence theorem, we deduce that

$$\begin{aligned}\lim_{n\to+\infty} D_{0^+}^{\alpha-1}u_n(t) &= \frac{1}{\Gamma(2-\alpha)} \lim_{n\to+\infty} \frac{d}{dt}\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)\frac{u_n(s)}{1+s^\alpha}ds \\ &= \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{1-\alpha}(1+s^\alpha)\frac{u(s)}{1+s^\alpha}ds = D^{\alpha-1}u(t).\end{aligned}$$

Thus $(Y, \|.\|_Y)$ is a Banach space. □

Before we proceed with the existence theory, we list some conditions:

$(H1)$ $0 < \beta\eta^{2\alpha-2} < \Gamma(2\alpha-1)$.

$(H2)$ The function $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous such that $\int_0^{+\infty} |f(s, 0, 0)|ds < +\infty$.

$(H3)$ There exists nonnegative functions $(1 + t^\alpha)g(t)$, $h(t) \in L^1[0, +\infty)$ such that $|f(t, x, \overline{x}) - f(t, y, \overline{y})| \leqslant g(t)|x - y| + h(t)|\overline{x} - \overline{y}|$ for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$ and $t \in [0, +\infty)$.

Lemma 3.2. *Let $e(t) \in L^1[0,+\infty)$. Under Hypothesis $(H1)$, the bvp*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + e(t) = 0, & t \in (0,+\infty), \\ u(0) = 0, & \lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \beta I_{0^+}^{\alpha-1} u(\eta), \end{cases} \tag{3.1}$$

has a unique solution given by

$$\begin{aligned} u(t) \;=\; & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s)ds + \frac{\Gamma(2\alpha-1)t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1) - \beta\eta^{2\alpha-2}\right)} \int_0^{+\infty} e(s)ds \\ & -\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1) - \beta\eta^{2\alpha-2}\right)} \int_0^{\eta} (\eta-s)^{2\alpha-2} e(s)ds. \end{aligned}$$

Proof. By Lemma 2.1 and since $D_{0^+}^{\alpha} u(t) + e(t) = 0$, we have

$$u(t) = -I_{0^+}^{\alpha} e(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \;\text{ for some constants }\; c_1,\, c_2 \in \mathbb{R}.$$

So the solution of (3.1) can be written as

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

From $u(0) = 0$ we infer that $c_2 = 0$. In addition

$$\begin{aligned} D_{0^+}^{\alpha-1} u(t) &= -D_{0^+}^{\alpha-1} I_{0^+}^{\alpha} e(t) + c_1 D_{0^+}^{\alpha-1}(t^{\alpha-1}) \\ &= -I_{0^+}^{1} e(t) + c_1 \Gamma(\alpha) \\ &= -\int_0^t e(s)ds + c_1 \Gamma(\alpha). \end{aligned}$$

So

$$\lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = -\int_0^{+\infty} e(s)ds + c_1 \Gamma(\alpha).$$

Moreover

$$\begin{aligned} I_{0^+}^{\alpha-1} u(t) &= -I_{0^+}^{\alpha-1} I_{0^+}^{\alpha} e(t) + c_1 I_{0^+}^{\alpha-1}(t^{\alpha-1}) \\ &= -I_{0^+}^{2\alpha-1} e(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)} t^{2\alpha-2} \\ &= -\frac{1}{\Gamma(2\alpha-1)} \int_0^t (t-s)^{2\alpha-2} e(s)ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)} t^{2\alpha-2}. \end{aligned}$$

Hence

$$I_{0^+}^{\alpha-1}u(\eta) = -\frac{1}{\Gamma(2\alpha-1)}\int_0^{\eta}(\eta-s)^{2\alpha-2}e(s)ds + c_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}\eta^{2\alpha-2},$$

together with $\lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \beta I_{0^+}^{\alpha-1}u(\eta)$,

$$c_1 = \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty}e(s)ds - \frac{\beta}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{\eta}(\eta-s)^{2\alpha-2}e(s)ds.$$

Therefore, the unique solution of fractional boundary value problem (3.1) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)ds + \frac{\Gamma(2\alpha-1)t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty}e(s)ds \\ &\quad -\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{\eta}(\eta-s)^{2\alpha-2}e(s)ds. \end{aligned}$$

□

Lemma 3.3. *Under Assumption* $(H1)$, *the solution of the bvp (3.1) can also be written as*

$$u(t) = \int_0^{+\infty} G(t,s)e(s)ds,$$

where $G(t,s)$ *is the Green's function*

$$G(t,s) = G_1(t,s) + G_2(t,s)$$

with

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leqslant s\leqslant t<+\infty, \\ t^{\alpha-1}, & 0\leqslant t\leqslant s<+\infty \end{cases}$$

and

$$G_2(t,s) = \frac{\beta t^{\alpha-1}}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\times\frac{1}{\Gamma(\alpha)}\begin{cases} \eta^{2\alpha-2}-(\eta-s)^{2\alpha-2}, & 0\leqslant s\leqslant \eta<+\infty, \\ \eta^{2\alpha-2}, & 0\leqslant \eta\leqslant s<+\infty. \end{cases}$$

Proof. The solution of problem (3.1) can be expressed as

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e(s)ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e(s)ds + \frac{\Gamma(2\alpha-1)t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty} e(s)ds \\
&\quad -\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{\eta}(\eta-s)^{2\alpha-2}e(s)ds \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t t^{\alpha-1}e(s)ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_t^{+\infty} t^{\alpha-1}e(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e(s)ds \\
&\quad +\frac{\Gamma(2\alpha-1)t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty} e(s)ds \\
&\quad -\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{\eta}(\eta-s)^{2\alpha-2}e(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left(t^{\alpha-1}-(t-s)^{\alpha-1}\right)e(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^{+\infty} t^{\alpha-1}e(s)ds \\
&\quad +\frac{\beta\eta^{2\alpha-2}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty} t^{\alpha-1}e(s)ds \\
&\quad -\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{\eta}(\eta-s)^{2\alpha-2}e(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left(t^{\alpha-1}-(t-s)^{\alpha-1}\right)e(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^{+\infty} t^{\alpha-1}e(s)ds \\
&\quad +\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{\eta}\left(\eta^{2\alpha-2}-(\eta-s)^{2\alpha-2}\right)e(s)ds \\
&\quad +\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_{\eta}^{+\infty}\eta^{2\alpha-2}e(s)ds \\
&= \int_0^{+\infty} G_1(t,s)e(s)ds + \int_0^{+\infty} G_2(t,s)e(s)ds.
\end{aligned}$$

□

Remark 3.1. *From $(H1)$ and the definition of $G_1(t,s)$ and $G_2(t,s)$, observe that*

(a) G_1 and G_2 are continuous and nonnegative functions on $[0,+\infty)\times[0,+\infty)$.

(b) $\frac{G_1(t,s)}{1+t^\alpha} \leqslant \frac{1}{\Gamma(\alpha)}$ for $(t,s)\in[0,+\infty)\times[0,+\infty)$.

(c) $\frac{G_2(t,s)}{1+t^\alpha} \leqslant \frac{\beta\eta^{2\alpha-2}}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2})}$ *for* $(t,s) \in [0,+\infty) \times [0,+\infty)$.

(d) $D_{0^+}^{\alpha-1}G_1(t,s) \leqslant 1,\ D_{0^+}^{\alpha-1}G_2(t,s) \leqslant \frac{\beta\eta^{2\alpha-2}}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}},\ t,\ s>0.$

Define the operators $T_1,\ T_2,\ T$ on Y by

$$\begin{aligned}(T_1u)(t) &= \int_0^{+\infty} G_1(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds,\\ (T_2u)(t) &= \int_0^{+\infty} G_2(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds,\\ (Tu)(t) &= (T_1u)(t)+(T_2u)(t).\end{aligned}$$

Bvp (1.1) has a solution u if and only if u solves the operator equation $u = Tu$.

We shall prove the existence of a fixed point of T. For this we verify that the operator T satisfies all conditions of Theorem 2.1.

Since the Arzela-Ascoli theorem fails to work in the space Y, we need a modified compactness criterion to prove that T_1 is compact.

Lemma 3.4. *([13]) Let $Z \subseteq Y$ be bounded set. Then Z is relatively compact on Y if for any $u \in Z$, $\frac{u(t)}{1+t^\alpha}$ and $D_{0^+}^{\alpha-1}u(t)$ are equicontinuous on any compact intervals of $[0,+\infty)$ and are equiconvergent at infinity.*

Definition 3.1. *$\frac{u(t)}{1+t^\alpha}$ and $D_{0^+}^{\alpha-1}u(t)$ are called equiconvergent at infinity if and only if for all $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that*

$$\left|\frac{u(t_1)}{1+t_1^\alpha} - \frac{u(t_2)}{1+t_2^\alpha}\right| < \varepsilon \text{ and } \left|D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)\right| < \varepsilon,$$

for any $t_1,\ t_2>\delta$ and $u \in Z$.

Let $\Omega_r = \{u \in Y : \quad \|u\|_Y < r\},\ (r>0)$ be the open ball of radius r in Y.

Lemma 3.5. *If $(H1)-(H3)$ hold, then $T(\overline{\Omega}_r)$ is bounded set.*

Proof. From $(H3)$ we have

$$|f(t,x,\overline{x})| \leqslant g(t)|x| + h(t)|\overline{x}| + |f(t,0,0)|, \quad \text{for } x,\ \overline{x} \in \mathbb{R} \text{ and } t \in [0,+\infty). \tag{3.2}$$

Furthermore, from Remark 3.1, we have for any $u \in \overline{\Omega}_r$,

$$\begin{aligned}
\sup_{t\geq 0}\left|\frac{(Tu)(t)}{1+t^\alpha}\right| &\leqslant \sup_{t\geq 0}\left|\int_0^{+\infty}\frac{G_1(t,s)}{1+t^\alpha}f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right| \\
&\quad + \sup_{t\geq 0}\left|\int_0^{+\infty}\frac{G_2(t,s)}{1+t^\alpha}f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\quad + \frac{\beta\eta^{2\alpha-2}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\left(r\int_0^{+\infty}\left((1+s^\alpha)g(s)+h(s)\right)ds\right. \\
&\quad \left. + \int_0^{+\infty}|f(s,0,0)|ds\right) < +\infty
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| &\leqslant \sup_{t\geq 0}\int_0^{+\infty}D_{0^+}^{\alpha-1}G_1(t,s)\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\quad + \sup_{t\geq 0}\int_0^{+\infty}D_{0^+}^{\alpha-1}G_2(t,s)\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\leqslant \int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\quad + \frac{\beta\eta^{2\alpha-2}}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\left(r\int_0^{+\infty}\left((1+s^\alpha)g(s)+h(s)\right)ds\right. \\
&\quad \left. + \int_0^{+\infty}|f(s,0,0)|ds\right) < +\infty.
\end{aligned}$$

So

$$\|Tu\|_Y < +\infty, \text{ for } u \in \overline{\Omega}_r.$$

□

Lemma 3.6. *If* $(H1)-(H3)$ *hold, then* $T_1 : \overline{\Omega}_r \to Y$ *is completely continuous.*

Proof. **Claim 1. The set $T_1(\overline{\Omega}_r)$ is bounded set.**

By definition of the operator T_1 and relation (3.2) we have that for any $u \in \overline{\Omega}_r$,

$$\begin{aligned}
\frac{|T_1 u(t)|}{1+t^\alpha} &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} (g(s)|u(s)| + h(s)|D_{0^+}^{\alpha-1}u(s)| + |f(s,0,0)|)ds \\
&\leqslant \frac{\|u\|_Y}{\Gamma(\alpha)}\int_0^{+\infty} \left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,0,0)|ds \\
&\leqslant \frac{r}{\Gamma(\alpha)}\int_0^{+\infty} \left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,0,0)|ds < +\infty.
\end{aligned}$$

Moreover

$$\begin{aligned}
|D_{0^+}^{\alpha-1}T_1 u(t)| &\leqslant \int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds \\
&\leqslant r\int_0^{+\infty} \left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds + \int_0^{+\infty} |f(s,0,0)|ds < +\infty.
\end{aligned}$$

Hence

$$\|T_1 u\|_Y < +\infty, \text{ for } u \in \overline{\Omega}_r.$$

Claim 2. T_1 is continuous. For this, let $u_n \to u$ as $n \to +\infty$ in $\overline{\Omega}_r$.

From Remark 3.1 and relation (3.2), we have the estimates:

$$\begin{aligned}
\left|\frac{(T_1 u_n)(t)}{1+t^\alpha} - \frac{(T_1 u)(t)}{1+t^\alpha}\right| &\leqslant \int_0^{+\infty} \frac{G_1(t,s)}{1+t^\alpha}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} (|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))| + |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|)ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} (g(s)|u_n(s)| + h(s)|D_{0^+}^{\alpha-1}u_n(s)| + |f(s,0,0)|)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} (g(s)|u(s)| + h(s)|D_{0^+}^{\alpha-1}u(s)| + |f(s,0,0)|)ds \\
&\leqslant \frac{\|u_n\|_Y}{\Gamma(\alpha)}\int_0^{+\infty} \left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,0,0)|ds \\
&\quad + \frac{\|u\|_Y}{\Gamma(\alpha)}\int_0^{+\infty} \left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,0,0)|ds.
\end{aligned}$$

Moreover

$$\left|\frac{(T_1u_n)(t)}{1+t^\alpha}-\frac{(T_1u)(t)}{1+t^\alpha}\right| \leqslant \frac{2}{\Gamma(\alpha)}\left(r\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right)<+\infty.$$

Using the continuity of f, we obtain that

$$|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0^+}^{\alpha-1}u(s))|\to 0, \text{ as } n\to+\infty,$$

which implies

$$\|T_1u_n-T_1u\|_X=\sup_{t\geq 0}\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}}-\frac{(T_1u)(t)}{1+t^{\alpha-1}}\right|\to 0,$$

uniformly as $n\to+\infty$.

As well as

$$\begin{aligned}\left|D_{0^+}^{\alpha-1}T_1u_n(t)-D_{0^+}^{\alpha-1}T_1u(t)\right| &\leqslant \int_0^{+\infty}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\leqslant 2r\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds+2\int_0^{+\infty}|f(s,0,0)|ds<+\infty.\end{aligned}$$

Using again the continuity of f, we get

$$\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}T_1u_n(t)-D_{0^+}^{\alpha-1}T_1u(t)\right|\to 0, \text{ uniformly as } n\to+\infty.$$

We conclude

$$\|T_1u_n-T_1u\|_Y\to 0, \text{ uniformly as } n\to+\infty, \text{ as claimed.}$$

Claim 3. $T_1:\overline{\Omega}_r\to Y$ **is relatively compact.** According to Claim 1, the set $T_1(\overline{\Omega}_r)$ is uniformly bounded. We show that functions from $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$ and functions from $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ are equicontinuous on any compact intervals of $[0,+\infty)$.

Let $I\subset[0,+\infty)$ be a compact interval, then for any $t_1,\ t_2\in I$ such that $t_1<t_2$ and for $u\in\overline{\Omega}_r$, we have

$$\begin{aligned}\left|\frac{(T_1u)(t_2)}{1+t_2^\alpha}-\frac{(T_1u)(t_1)}{1+t_1^\alpha}\right| &\leqslant \int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_1^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\leqslant \int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_2^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\quad+\int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_1^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds.\end{aligned}$$

Moreover

$$
\begin{aligned}
&\int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_2^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\quad\leqslant \int_0^{t_1}\left|\frac{G_1(t_2,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_2^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\qquad+\int_{t_1}^{t_2}\left|\frac{G_1(t_2,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_2^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\qquad+\int_{t_2}^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_2^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}+(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\right|\left(\left(1+s^\alpha\right)g(s)r+h(s)r\right.\\
&\qquad\qquad\left.+|f(s,0,0)|\right)ds\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\right|\left(\left(1+s^\alpha\right)g(s)r+h(s)r+|f(s,0,0)|\right)ds\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{1+t_2^\alpha}\right|\left(\left(1+s^\alpha\right)g(s)r+h(s)r+|f(s,0,0)|\right)ds,
\end{aligned}
$$

which converges to 0 uniformly as $|t_1-t_2|\to 0$.

We have also the estimates:

$$
\begin{aligned}
&\int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_2^\alpha}-\frac{G_1(t_1,s)}{1+t_1^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\qquad\leqslant \int_0^{+\infty}\left|\frac{t_1^\alpha-t_2^\alpha}{1+t_2^\alpha}\right|\left|\frac{G_1(t_1,s)}{1+t_1^\alpha}\right|\left(\left(1+s^\alpha\right)g(s)r+h(s)r+|f(s,0,0)|\right)ds\\
&\qquad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\left|\frac{t_1^\alpha-t_2^\alpha}{1+t_2^\alpha}\right|\left(\left(1+s^\alpha\right)g(s)r+h(s)r+|f(s,0,0)|\right)ds\to 0,
\end{aligned}
$$

uniformly as $|t_1-t_2|\to 0$. Similarly

$$
\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_1u(t_2)-D_{0^+}^{\alpha-1}T_1u(t_1)\right| &\leqslant \int_{t_1}^{t_2}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds\\
&\leqslant \int_{t_1}^{t_2}\left(\left(1+s^\alpha\right)g(s)r+h(s)r+|f(s,0,0)|\right)ds,
\end{aligned}
$$

which converges to 0 uniformly as $|t_1-t_2|\to 0$. Then, for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$
\left|\frac{(T_1u)(t_2)}{1+t_2^\alpha}-\frac{(T_1u)(t_1)}{1+t_1^\alpha}\right|<\varepsilon \quad\text{and}\quad \left|D_{0^+}^{\alpha-1}T_1u(t_2)-D_{0^+}^{\alpha-1}T_1u(t_1)\right|<\varepsilon,
$$

for all $u \in \overline{\Omega}_r$, as $|t_2 - t_1| < \delta,\ t_1,\ t_2 \in I$.

This proves that the functions belonging to $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$ and the functions belonging to $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ are locally equicontinuous on $[0,+\infty)$.

Next we show that functions from $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$ and functions from $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ are equiconvergent at infinity.

For any $u \in \overline{\Omega}_r$, we have

$$\int_0^{+\infty} \left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right| ds < +\infty.$$

Considering condition $(H2)$ and relation (3.2), for given $\varepsilon > 0$, there exists a constant $L > 0$ such that

$$\int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s))| ds < \varepsilon.$$

On the other hand, since $\lim\limits_{t\to+\infty} \frac{t^{\alpha-1}}{1+t^\alpha} = 0$, there exists a constant $\delta_1 > 0$ such that for any $t_1,\ t_2 > \delta_1$,

$$\left|\frac{t_i^{\alpha-1}}{1+t_i^\alpha}\right| < \frac{\varepsilon}{2},\ (i=1,\,2) \text{ and } \left|\frac{t_2^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{\alpha-1}}{1+t_1^\alpha}\right| < \varepsilon.$$

In the same way and in view of $\lim\limits_{t\to+\infty} \frac{(t-L)^{\alpha-1}}{1+t^\alpha} = 0$, there exists a constant $\delta_2 > L > 0$ such that for any $t_1,\ t_2 > \delta_2$ and $0 \leqslant s \leqslant L$,

$$\left|\frac{(t_i-s)^{\alpha-1}}{1+t_i^\alpha}\right| < \frac{\varepsilon}{2},\ (i=1,\,2) \text{ and } \left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right| < \varepsilon.$$

Select $\delta > \max\{\delta_1,\ \delta_2\}$. Then for any $t_1,\ t_2 > \delta,\ t_1 < t_2$, we have the estimates:

$$\begin{aligned}
\left|\frac{(T_1u)(t_2)}{1+t_2^\alpha} - \frac{(T_1u)(t_1)}{1+t_1^\alpha}\right| &\leqslant \int_0^{+\infty} \left|\frac{G_1(t_2,s)}{1+t_2^\alpha} - \frac{G_1(t_1,s)}{1+t_1^\alpha}\right| |f(s,u(s),D_{0^+}^{\alpha-1}u(s))| ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} \left|\frac{t_2^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{\alpha-1}-(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\right. \\
&\quad |f(s,u(s),D_{0^+}^{\alpha-1}u(s))| ds \\
&\quad + \int_{t_1}^{t_2} \left|\frac{t_2^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{\alpha-1}}{1+t_1^\alpha}\right| |f(s,u(s),D_{0^+}^{\alpha-1}u(s))| ds \\
&\quad \left. + \int_{t_2}^{+\infty} \left|\frac{t_2^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{\alpha-1}}{1+t_1^\alpha}\right| |f(s,u(s),D_{0^+}^{\alpha-1}u(s))| ds\right).
\end{aligned}$$

Moreover

$$\begin{aligned}\left|\frac{(T_1u)(t_2)}{1+t_2^\alpha}-\frac{(T_1u)(t_1)}{1+t_1^\alpha}\right| &\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^L\left|\frac{t_2^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{\alpha-1}-(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\right.\\ &\quad |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\quad +\int_L^{+\infty}\left|\frac{t_2^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{\alpha-1}-(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\\ &\quad |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\quad +\int_L^{+\infty}\left|\frac{t_2^{\alpha-1}-(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{\alpha-1}}{1+t_1^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\quad \left.+\int_L^{+\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{\alpha-1}}{1+t_1^\alpha}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\right)\\ &\leqslant \frac{1}{\Gamma(\alpha)}\left(\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|2L\varepsilon+\frac{9}{2}\varepsilon^2\right)\end{aligned}$$

and

$$\begin{aligned}\left|D_{0^+}^{\alpha-1}T_1u(t_2)-D_{0^+}^{\alpha-1}T_1u(t_1)\right| &\leqslant \int_{t_1}^{t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\leqslant \int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds<\varepsilon\end{aligned}$$

which implies that the functions from $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$ and the functions from $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ are equiconvergent at infinity.

By Lemma 3.4, we deduce $T_1:\overline{\Omega}_r\to Y$ is relatively compact, ending the proof of the Lemma. □

Lemma 3.7. *If* $(H1)-(H3)$ *hold and that*

$(H4)$ $\frac{\beta\eta^{2\alpha-2}}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\int_0^{+\infty}\left((1+s^\alpha)\,g(s)+h(s)\right)ds<\Gamma(\alpha).$

Then $T_2:\overline{\Omega}_r\to Y$ *is a contractive mapping.*

Proof. According to Remark 3.1 and condition $(H3)$, we have

$$\begin{aligned}\left|\frac{T_2u(t)}{1+t^\alpha}-\frac{T_2v(t)}{1+t^\alpha}\right| &\leqslant \int_0^{+\infty}\frac{G_2(t,s)}{1+t^\alpha}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))-f(s,v(s),D_{0^+}^{\alpha-1}v(s))|ds\\ &\leqslant \frac{\beta\eta^{2\alpha-2}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty}\left(\sup\left|\frac{u(s)-v(s)}{1+s^\alpha}\right|(1+s^\alpha)g(s)\right.\\ &\quad \left.+\sup\left|D_{0^+}^{\alpha-1}u(s)-D_{0^+}^{\alpha-1}v(s)\right|h(s)\right)ds,\ \text{ for all } u,\,v\in\overline{\Omega}_r \text{ and } t\geqslant 0.\end{aligned}$$

Hence

$$\left|\frac{T_2u(t)}{1+t^\alpha}-\frac{T_2v(t)}{1+t^\alpha}\right| \leqslant \frac{\beta\eta^{2\alpha-2}\|u-v\|_Y}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds$$

as well as

$$\left|D_{0^+}^{\alpha-1}T_2u(t)-D_{0^+}^{\alpha-1}T_2v(t)\right| \leqslant \frac{\beta\eta^{2\alpha-2}\|u-v\|_Y}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds.$$

Since $\frac{1}{\Gamma(\alpha)}\geqslant 1$, then

$$\|T_2u-T_2v\|_Y \leqslant \frac{\beta\eta^{2\alpha-2}}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\left(\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds\right)\|u-v\|_Y.$$

From $(H4)$, we infer that T_2 is contractive. □

4 Main results

Theorem 4.1. *Further to Assumptions* $(H1)-(H4)$, *assume that*

$(H5)$ *There exists* $\rho>0$ *such that*

$$\frac{\rho\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}{\Gamma(2\alpha-1)\left(\rho\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right)}>\frac{1}{\Gamma(\alpha)}.$$

Then problem (1.1) has at least one solution.

Proof. Consider the parameterized bvp

$$\begin{cases} D_{0^+}^{\alpha}u(t)+\lambda f(t,u(t),D_{0^+}^{\alpha-1}u(t))=0, & t>0,\\ u(0)=0, \qquad \lim\limits_{t\to+\infty}D_{0^+}^{\alpha-1}u(t)=\beta I_{0^+}^{\alpha-1}u(\eta), \end{cases} \tag{4.1}$$

for $\lambda\in(0,\ 1)$. Solving problem (4.1) is equivalent to solving the fixed point of equation $u=\lambda Tu$.

Let

$$\Omega_\rho=\{u\in Y:\quad \|u\|_Y<\rho\},$$

for some positive constant ρ. From Lemma 3.5, the set $T(\overline{\Omega}_\rho)$ is bounded and by Lemma 3.6, the operator $T_1 : \overline{\Omega}_\rho \to Y$ is completely continuous while Lemma 3.7 implies that the operator $T_2 : \overline{\Omega}_\rho \to Y$ is contractive.

So it remains to prove that $u \neq \lambda T u$ for $u \in \partial\Omega_\rho$ and $\lambda \in (0,\ 1)$.

Arguing by contradiction, assume that there exists $u \in \partial\Omega_\rho$ with $u = \lambda T u$. The same way in the argument as Lemma 3.5 we have for $\lambda \in (0,\ 1)$,

$$\begin{aligned}
\sup_{t\geq 0}\left|\frac{u(t)}{1+t^\alpha}\right| &= \sup_{t\geq 0}\left|\frac{(\lambda T u)(t)}{1+t^\alpha}\right| \\
&\leqslant \sup_{t\geq 0}\left|\frac{(T u)(t)}{1+t^\alpha}\right| \\
&\leqslant \sup_{t\geq 0}\left|\int_0^{+\infty}\frac{G_1(t,s)}{1+t^\alpha} f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right| \\
&\quad + \sup_{t\geq 0}\left|\int_0^{+\infty}\frac{G_2(t,s)}{1+t^\alpha} f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right| \\
&\leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\left(\rho\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds\right. \\
&\quad \left. + \int_0^{+\infty}|f(s,0,0)|ds\right)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}u(t)\right| &= \sup_{t\geq 0}\left|\lambda D_{0^+}^{\alpha-1}Tu(t)\right| \\
&\leqslant \sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \\
&\leqslant \sup_{t\geq 0}\int_0^{+\infty} D_{0^+}^{\alpha-1}G_1(t,s)\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\quad + \sup_{t\geq 0}\int_0^{+\infty} D_{0^+}^{\alpha-1}G_2(t,s)\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s))\right|ds \\
&\leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\left(\rho\int_0^{+\infty}\left(\left(1+s^\alpha\right)g(s)+h(s)\right)ds\right. \\
&\quad \left. + \int_0^{+\infty}|f(s,0,0)|ds\right).
\end{aligned}$$

Hence

$$\|u\|_Y \leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\left(\rho\int_0^{+\infty}\left(\left(1+s^{\alpha}\right)g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right)$$

and thus

$$\rho \leqslant \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}\left(\rho\int_0^{+\infty}\left(\left(1+s^{\alpha}\right)g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right).$$

This implies that

$$\frac{\rho\Gamma(\alpha)\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}{\Gamma(2\alpha-1)\left(\rho\int_0^{+\infty}\left(\left(1+s^{\alpha}\right)g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right)} \leqslant 1,$$

contradicting condition $(H5)$. With Theorem 2.1, we conclude that bvp (1.1) has at least one solution. □

5 Examples

Example 5.1. *Consider the bvp on infinite interval*

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t)+\frac{u(t)}{(28+t)^2\left(1+\sqrt{t^3}\right)}+\frac{D_{0^+}^{\frac{1}{2}}u(t)}{3e^t-1}+e^{-t}=0, & t>0,\\ u(0)=0, \qquad \lim\limits_{t\to+\infty}D_{0^+}^{\frac{1}{2}}u(t)=\frac{1}{2}I_{0^+}^{\frac{1}{2}}u(1). \end{cases} \tag{5.1}$$

In this case, $\alpha=\frac{3}{2},\ \Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2},\ \Gamma(2\alpha-1)=\Gamma(2)=1,\ \beta=\frac{1}{2},\ \eta=1.$ *Let*

$$f(t,x,y)=\frac{x}{(28+t)^2\left(1+\sqrt{t^3}\right)}+\frac{y}{3e^t-1}+e^{-t}$$

and

$$g(t)=\frac{1}{(28+t)^2\left(1+\sqrt{t^3}\right)}, \quad h(t)=\frac{1}{3e^t-1}.$$

Choose

$$\rho>\frac{1}{\frac{7\sqrt{\pi}-1}{28}-\ln(\frac{3}{2})}.$$

We have

$$\int_0^{+\infty}\left(1+s^{\frac{3}{2}}\right)g(s)ds = \frac{1}{28}<+\infty,\ \int_0^{+\infty}h(s)ds=\ln\left(\frac{3}{2}\right)<+\infty,$$
$$\int_0^{+\infty}|f(s,0,0)|ds = \int_0^{+\infty}e^{-s}ds=1<+\infty.$$

Then

$(H1)$ $0<\beta\eta^{2\alpha-2}<\Gamma(2\alpha-1)$.

$(H2)$ $f:[0,+\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is continuous and* $\int_0^{+\infty}|f(s,0,0)|ds<+\infty$.

$(H3)$ $|f(t,x,\overline{x})-f(t,y,\overline{y})|\leqslant g(t)|x-y|+h(t)|\overline{x}-\overline{y}|$, *for all* x, y, $\overline{x}$, $\overline{y}\in\mathbb{R}$ *and* $t\in[0,+\infty)$.

$(H4)$

$$\frac{\beta\eta^{2\alpha-2}}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\int_0^{+\infty}\left(\left(1+s^{\alpha}\right)g(s)+h(s)\right)ds = \frac{\frac{1}{2}}{1-\frac{1}{2}}\left(\frac{1}{28}+\ln\left(\frac{3}{2}\right)\right) < \Gamma(\frac{3}{2}).$$

$(H5)$

$$\frac{\rho\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}{\Gamma(2\alpha-1)\left(\rho\int_0^{+\infty}\left(\left(1+s^{\alpha}\right)g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right)} = \frac{\frac{1}{2}\rho}{\left(\frac{1}{28}+\ln\left(\frac{3}{2}\right)\right)\rho+1} = \frac{\frac{1}{2}}{\frac{1}{28}+\ln\left(\frac{3}{2}\right)+\frac{1}{\rho}} > \frac{1}{\Gamma(\frac{3}{2})}.$$

Hence, all conditions of Theorem 4.1 are satisfied, we deduce that bvp (5.1) has at least one solution.

Example 5.2. *Consider the bvp on infinite interval*

$$\begin{cases} D_{0^+}^{\frac{5}{4}}u(t)-\frac{|u(t)|e^{-t}}{6(1+|u(t)|)\left(1+\sqrt[4]{t^5}\right)}+\frac{1}{\left(1+\left|D_{0^+}^{\frac{1}{4}}u(t)\right|\right)(5+t)^2}+\frac{4}{5}e^{-t}=0, & t>0,\\ u(0)=0, \quad \lim_{t\to+\infty}D_{0^+}^{\frac{1}{4}}u(t)=\frac{3}{2}I_{0^+}^{\frac{1}{4}}u(\frac{1}{9}). \end{cases} \tag{5.2}$$

In this case, $\alpha = \frac{5}{4},\ \Gamma(\frac{5}{4}) \approx 0.906402,\ \Gamma(2\alpha-1) = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2},\ \beta = \frac{3}{2},\ \eta = \frac{1}{9}$. *Let*

$$f(t,x,y) = -\frac{e^{-t}}{6\left(1+\sqrt[4]{t^5}\right)} \times \frac{|x|}{1+|x|} + \frac{1}{(5+t)^2(1+|y|)} + \frac{4}{5}e^{-t},$$

$$g(t) = \frac{e^{-t}}{6\left(1+\sqrt[4]{t^5}\right)}, \quad h(t) = \frac{1}{(5+t)^2},$$

and select

$$\rho > \frac{30\sqrt{\pi}}{30\Gamma(\frac{5}{4})(\sqrt{\pi}-1) - 11\sqrt{\pi}}.$$

Then

$$\begin{aligned}\int_0^{+\infty}\left(1+\sqrt[4]{s^5}\right)g(s)ds &= \frac{1}{6} < +\infty, \int_0^{+\infty} h(s)ds = \frac{1}{5} < +\infty,\\ \int_0^{+\infty}|f(s,0,0)|ds &= \int_0^{+\infty}\left(\frac{1}{(5+s)^2} + \frac{4}{5}e^{-s}\right)ds = 1 < +\infty.\end{aligned}$$

Moreover

$$\begin{aligned}|f(t,x,\overline{x}) - f(t,y,\overline{y})| &\leqslant \frac{e^{-t}}{6\left(1+\sqrt[4]{t^5}\right)}\left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right|\\ &\quad + \frac{1}{(5+t)^2}\left|\frac{1}{1+|\overline{x}|} - \frac{1}{1+|\overline{y}|}\right|\\ &\leqslant \frac{e^{-t}}{6\left(1+\sqrt[4]{t^5}\right)} \times \frac{|x-y|}{(1+|x|)(1+|y|)}\\ &\quad + \frac{1}{(5+t)^2} \times \frac{|\overline{x}-\overline{y}|}{(1+|\overline{x}|)(1+|\overline{y}|)}\\ &\leqslant g(t)|x-y| + h(t)|\overline{x}-\overline{y}|,\end{aligned}$$

for all $x,\ y,\ \overline{x},\ \overline{y} \in \mathbb{R}$ *and* $t \in [0,+\infty)$.

Then

$(H1)$ $0 < \beta\eta^{2\alpha-2} < \Gamma(2\alpha-1)$.

$(H2)$ $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and* $\int_0^{+\infty}|f(s,0,0)|ds < +\infty$.

$(H3)$ $|f(t,x,\overline{x}) - f(t,y,\overline{y})| \leqslant g(t)|x-y| + h(t)|\overline{x}-\overline{y}|$ *for all* $x,\ y,\ \overline{x},\ \overline{y} \in \mathbb{R}$ *and*

$t \in [0, +\infty)$.

$(H4)$

$$\frac{\beta\eta^{2\alpha-2}}{\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}}\int_0^{+\infty}\left((1+s^{\alpha})\,g(s)+h(s)\right)ds = \frac{11}{30\left(\sqrt{\pi}-1\right)} < \Gamma(\frac{5}{4}).$$

$(H5)$

$$\begin{aligned}\frac{\rho\left(\Gamma(2\alpha-1)-\beta\eta^{2\alpha-2}\right)}{\Gamma(2\alpha-1)\left(\rho\int_0^{+\infty}\left((1+s^{\alpha})\,g(s)+h(s)\right)ds+\int_0^{+\infty}|f(s,0,0)|ds\right)} &= \frac{\left(\sqrt{\pi}-1\right)\rho}{\sqrt{\pi}\left(\frac{11}{30}\rho+1\right)}\\ &= \frac{\sqrt{\pi}-1}{\sqrt{\pi}\left(\frac{11}{30}+\frac{1}{\rho}\right)}\\ &> \frac{1}{\Gamma(\frac{5}{4})}.\end{aligned}$$

Hence, all conditions of Theorem 4.1 are satisfied, which guarantee that bvp (5.2) has at least one solution.

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A JUSTIFICATION OF TWO-DIMENSIONAL NONLINEAR SHELLS MODEL WITH VON KÁRMÁN BOUNDARY CONDITIONS

MARWA LEGOUGUI

Abstract. In this work, using the method of asymptotic expansions with the thickness as the "small" parameter, we show that the three-dimensional for a nonlinearly elastic shells of Saint Venant-Kirchhoff material with boundary conditions of von Kármán's type, written in curvilinear coordinates reduces to two-dimensional von Kármán model.

1. Introduction

The von Kármán equations are two-dimensional model for a nonlinearly elastic plate subjected to boundary conditions of von Kármáns type. They were initially proposed by von Kármán [7], which originating from continuum mechanics and play an important role in applied mathematics. Next, these equations are extended to Marguerre- von Kármán equations for a nonlinearly elastic shallow shell by Marguerre [6]. Then Ciarlet [1] and Ciarlet and Paumier [2] justified the both of previous models by formal asymptotic methods.

The asymptotic methods can be used for justifying the two-dimensional models of elastic plates and shells starting from the three-dimensional models. Numerous works have been devoted to plates and shells in static case (see, e.g., [3, 4]. For dynamical case, we refer to Ghezal and Chacha [5].

A natural question arises as: How to extend the von Kármán and Marguerre-von Kármán equations to the more general geometry of a shell?

2. Three-dimensional problem

Throughout this paper, we use the following conventions and notations: Greek indices (except for ε), belong to the set $\{1,2\}$, while Latin indices belong to the set $\{1,2,3\}$, the symbols of differentiation $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_i^\varepsilon = \frac{\partial}{\partial x_i^\varepsilon}$, $\hat{\partial}_i^\varepsilon = \frac{\partial}{\partial \hat{x}_i^\varepsilon}$, δ_{ij} the Kronecker symbols. The summation convention with respect to repeated indices is systematically used.

Consider a nonlinearly elastic shell with middle surface $S = \theta(\bar{\omega})$ and thickness $2\varepsilon > 0$, its constituting material is a Saint Venant-Kirchhoff material with Lam constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$, where ω is a domain in $\mathbb{R}^2$ with a boundary γ, and $\theta : \bar{\omega} \longrightarrow E^3$ is a smooth enough injective immersion, such that the two vectors $a_\alpha(y) = \partial_\alpha \theta(y)$ are linearly independent at all points $y \in \bar{\omega}$, which form the covariant basis of the tangent plane to the surface $S = \theta(\bar{\omega})$.

We define the mapping $\Theta : \bar{\Omega}^\varepsilon \longrightarrow \mathbb{R}^3$ as follow:

$$\Theta(x^\varepsilon) = \theta(y) + x_3^\varepsilon a_3(y),\ \forall (y, x_3) \in \bar{\Omega}^\varepsilon,$$

2000 *Mathematics Subject Classification.* 74B20, 74K25, 74G10.

Key words and phrases. Nonlinear elasticity; shell theory; von Kármán conditions; asymptotic analysis.

where

$$a_3(y) = a^3(y) = \frac{a_1 \wedge a_2}{|a_1 \wedge a_2|}.$$

The mapping Θ is assumed to be an immersion, the three vectors $g_i(x) = \partial_i \Theta(x)$, which are linearly independent at all points $x \in \bar{\Omega}$, thus form the covariant basis at $\hat{x} = \Theta(x) \in \bar{\hat{\Omega}}$.

For each $\varepsilon > 0$, we define the sets:

$$\bar{\Omega}^\varepsilon = \bar{\omega} \times [-\varepsilon, \varepsilon],\ \Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon],\ \Gamma_\pm^\varepsilon = \omega \times \{\pm\varepsilon\}.$$

The shell is subjected to applied body forces in its interior $\hat{\Omega}^\varepsilon = \Theta(\Omega^\varepsilon)$, of density $(\hat{f}_i^\varepsilon) : \hat{\Omega}^\varepsilon \longrightarrow \mathbb{R}^3$, to applied surface forces on the upper and the lower faces $\hat{\Gamma}_\pm^\varepsilon = \Theta(\Gamma_\pm^\varepsilon)$, of density $(\hat{l}_i^\varepsilon) : \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}^\varepsilon \longrightarrow \mathbb{R}^3$, and to horizontal forces on the lateral face $\hat{\Gamma}_0^\varepsilon = \Theta(\Gamma_0^\varepsilon)$, which we are given the averaged density $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0) : \Theta(\gamma) \longrightarrow \mathbb{R}^3$, after integration across the thickness of the shell. The displacement verifies specific conditions on the lateral face, in that only horizontal displacements are allowed along every vertical segment of the lateral face.

We define the espace

$$\mathbf{W}(\hat{\Omega}^\varepsilon) = \left\{ \hat{v}^\varepsilon = (\hat{v}_i^\varepsilon) \in W^{1,4}(\hat{\Omega}^\varepsilon; \mathbb{R}^3);\ \hat{v}_\alpha^\varepsilon \text{ is independent of } \hat{x}_3^\varepsilon, \hat{v}_3^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon \right\}.$$

The three-dimensional von Kármán shell problem reads:

$$\mathbf{P}(\hat{\Omega}^\varepsilon) \begin{cases} \text{Find } \hat{u}^\varepsilon = (\hat{u}_i^\varepsilon) \in \mathbf{W}(\hat{\Omega}^\varepsilon), \text{ such that,} \\ -\hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon = \hat{l}_i^\varepsilon \text{ on } \hat{\Gamma}_-^\varepsilon \cup \hat{\Gamma}_+^\varepsilon, \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} (\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon)\nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \text{ on } \Theta(\gamma), \end{cases}$$

where

$$\hat{\sigma}_{ij}^\varepsilon = \lambda^\varepsilon \hat{E}_{pp}^\varepsilon(\hat{u}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon),$$
$$\hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{u}_m^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_m^\varepsilon).$$

3. Two-dimensional models

We define the covariant components u_m^ε of the displacement field by:

$$\hat{u}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = u_m^\varepsilon(x^\varepsilon) g^{m,\varepsilon}(x^\varepsilon),\ \forall \hat{x}^\varepsilon = \Theta(x^\varepsilon) \in \{\bar{\hat{\Omega}}^\varepsilon\},$$

where $(\hat{e}^i)$ denotes the canonical basis of $\mathbb{R}^3$ and $(g^{m,\varepsilon}(x^\varepsilon))$ is the contravariant basis at the point $\hat{x^\varepsilon}$.

The covariant basis $(g_i^\varepsilon(x^\varepsilon))$, is given by

$$g_i^\varepsilon(x^\varepsilon) = \partial_i^\varepsilon \Theta(x^\varepsilon),\ \forall x^\varepsilon \in \Omega^\varepsilon.$$

The Christoffel symbols and the covariant and contravariant components of the metric tensor, defined by

$$\Gamma_{ij}^{k,\varepsilon} = \partial_i^\varepsilon g_j^\varepsilon . g^{k,\varepsilon},\ g_{ij}^\varepsilon = g_i^\varepsilon . g_j^\varepsilon\ g^{ij,\varepsilon} = g^{i,\varepsilon} . g^{j,\varepsilon}.$$

We define the contravariant components of the applied body and surface forces by

$$\hat{f}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = f^{i,\varepsilon}(x^\varepsilon) g_i^\varepsilon(x^\varepsilon),\ \forall x^\varepsilon \in \Omega^\varepsilon,$$

$$\hat{l}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = l^{i,\varepsilon}(x^\varepsilon)g_i^\varepsilon(x^\varepsilon),\ \forall x^\varepsilon \in \Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon.$$

Let $\bar{h}_\alpha^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \Theta$ on γ, defining the contravariant components $h^{i,\varepsilon}$ on γ by

$$\bar{h}_i^\varepsilon(y)e^i = h^{i,\varepsilon}(y)g_i^\varepsilon(y, x_3).$$

We define the following space

$$\mathbf{W}(\Omega) = \{v = (v_i) \in W^{1,4}(\Omega;\mathbb{R}^3); v_\alpha \text{ is independent of } x_3, v_3 = 0 \text{ on } \Gamma_0\}.$$

Assume that the scaled unknown $u(\varepsilon) = u_i^\varepsilon$ admits a formal asymptotic expansion of the form

$$u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...,$$

with

$$u^0 \in \mathbf{W}(\Omega) \quad and \quad u^p \in W^{1,4}(\Omega),\ \forall p \geq 1.$$

The components of the applied forces are of the form

$$f^{i,\varepsilon}(x^\varepsilon) = f^{i,0}(x),\ l^{i,\varepsilon}(x^\varepsilon) = \varepsilon l^{i,1}(x),\ h^{\alpha,\varepsilon}(y) = h^{\alpha,0}(y),$$

where the functions $f^{i,0} \in L^2(\Omega)$ and $l^{i,1} \in L^2(\Gamma_+ \cup \Gamma_-)$ and $h^{\alpha,0} \in L^2(\gamma)$ are independent of ε.

We now give the main results of this work

Theorem 3.1. *The leading term u^0 is independent of the transverse variable x_3 and it can be identified with ζ^0, which satisfies the following two-dimensional variational problem:*

$$\zeta^0 \in \mathbf{W}(\omega) = \{\eta \in W^{1,4}(\omega); \eta = 0 \ \ on\ \gamma\},$$

$$\int_\omega a^{\alpha\beta\sigma\tau} E^0_{\sigma\|\tau} F^0_{\alpha\|\beta}(\eta)\sqrt{a}dy = \int_\omega P^{i,0}\eta_i\sqrt{a}dy + 2\int_\gamma h^{\alpha,0}\eta_\alpha\sqrt{a_{\alpha\beta}(f(t))\frac{df^\alpha}{dt}(t)\frac{df^\beta}{dt}(t)}dt,$$

for all $\eta = (\eta_i) \in \mathbf{W}(\omega)$.
Where

$$E^0_{\alpha\|\beta} = \frac{1}{2}(\zeta^0_{\alpha\|\beta} + \zeta^0_{\beta\|\alpha} + a^{mn}\zeta^0_{m\|\alpha}\zeta^0_{n\|\beta}),$$

$$F^0_{\alpha\|\beta}(\eta) = \frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\{\zeta^0_{m\|\alpha}\eta_{n\|\beta} + \zeta^0_{n\|\beta}\eta_{m\|\alpha}\}),$$

$$\eta_{\alpha\|\beta} := \partial_\beta\eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,$$

$$\eta_{3\|\beta} := \partial_\beta\eta_3 + b^\sigma_\beta\eta_\sigma,$$

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),$$

$$P^{i,0} := \int_{-1}^1 f^{i,0} + l^{i,1}_- + l^{i,1}_+ \quad and \quad l^{i,1}_\pm := l^{i,1}(.,\pm 1).$$

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Best approximate solutions of nonlinear perturbed differential equations

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Abstract

In this work, we interested for Van Der pol equation in their general form as

$$\ddot{x} + \epsilon \left(b\dot{x}^2 + ax^2 - 1 \right) \dot{x} + x = 0. \quad x(0) = A \text{ and } \dot{x}(0) = 0, \tag{1}$$

with $a, b \in \mathbb{R},\ A > 0,\ 0 < \epsilon << 1$.

we studied approximate solutions of the equations of Van Der Pol oscillator (1) in several methods and compared them with the real solution as well as with each other, in order to search for the best method to provide the best approximate solution.

AMS subject classification: 76M45, 41A60, 35B10.
Keywords: Perturbations, Asymptotic approximations, Van Der Pol equation, Lindestedt-Poincar method, averaging method, renormalization group method.

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The Stochastic control theory in the G-frame work
La G- Théorie de Contrôle Stochastique

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Résumé
On s'intéresse aux problèmes de contrôle stochastique, on établit l'existence d'un contrôle optimal relaxé pour une équation différentielle stochastique gouvernée par un G- mouvement brownien. On démontre les conditions de régularité vérifiées par ce contrôle.

Introduction
l'objectif de ce travail est d'étudier le problème de contrôle stochastique partant d'une équation différentielle stochastique gouvernée par un G-mouvement brownien. L'intérêt des problèmes de contrôle réside dans l'optimisation d'un certain critère de performance appelé fonction coût à l'aide du contrôle optimal. Il existe de nombreux domaines typiques tels que les conditions météorologiques, le climat et les marchés financiers, où l'information est soumise à l'incertitude. Par exemple dans le problème de choix de portefeuille optimal en finance, où le processus de volatilité et de prime de risque sont inconnus et difficiles à estimer à partir de données fiables, nous devons considérer une famille de différents modèles ou scénarios.

Résultats principaux
.On établit l'existence d'un contrôle optimal relaxé pour une équation différentielle stochastique gouvernée par un G-mouvement brownien.

L'outil essentiel de notre contribution est le G Chattering lemme, qui nous a permis de généraliser le célébre Chattering lemme classique au cas d'un éspace d'éspérance sous linéaire.
On commence par définir l'espace des contrôles relaxés à l'aide des outils récents du G-calcul stochastique. Par la suite, on démontre l'existence d'un contrôle optimal relaxé vérifiant toutes les conditions de régularité.

Conclusion

On a prouvé l'existence d'un G-contrôle optimal relaxé dans le cadre de l'espérance non linéaire, satisfaisant les conditions de régularité nécessaires.

On a démontré l'existence en se basant sur le G-Chattering lemme, qui a été la clé de nos résultats.
On utilisant l'approximation des trajectoires, on a prouvé que d'une part chaque G-contrôle relaxé est une limite d'une suite de contrôles stricts bien définis et d'autre part, que chaque processus de diffusion lié au G-contrôle relaxé est une limite au sens fort d'une suite de processus de diffusion associés aux G-contrôles stricts.

Mots clés: G-équation différentielle stochastique, fonctionnelle coût, contrôle optimal relaxé.
2010 Mathematics Subject Classification. 60H10, 6007, 49N90H.

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ASYMPTOTIC CONVERGENCE OF A GENERALIZED NON-NEWTONIAN FLUID WITH TRESCA BOUNDARY CONDITIONS

ABDELKADER SAADALLAH

Abstract. The goal of this article is to study the asymptotic analysis of an incompressible Herschel-Bulkley fluid in a thin domain with Tresca boundary conditions. The yield stress and the constant viscosity are assumed to vary with respect to the thin layer parameter ϵ. Firstly, the problem statement and variational formulation are formulated. We then obtained the existence and the uniqueness result of a weak solution and the estimates for the velocity field and the pressure independently of the parameter ϵ. Finally, we give a specific Reynolds equation associated with variational inequalities and prove the uniqueness.

2010 Mathematics Subject Classification. 35R35; 76F10; 78M35.

Keywords and phrases. Asymptotic approach; Herschel-Bulkley fluid; Reynolds equation; Tresca law.

1. Define the problem

Let ω be fixed region in plan $x = (x_1, x_2) \in \mathbb{R}^2$. We suppose that ω has a *Lipschitz* boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x)$ where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < h_* < h(x) < h^*$ for all $(x, 0) \in \omega$ and Γ_L^ε the lateral surface. We denote by Ω^ε the domain of the flow :

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3 : (x, 0) \in \omega, 0 < x_3 < \varepsilon h(x)\}.$$

The boundary of Ω^ε is Γ^ε. We have $\Gamma^\varepsilon = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup \bar{\omega}$ where $\bar{\Gamma}_L$ is the lateral boundary.

- The law of conservation of momentum is given by the equation

$$u^\varepsilon \nabla u^\varepsilon = \operatorname{div}(\sigma^\varepsilon) + f^\varepsilon \text{ in } \Omega^\varepsilon, \tag{1}$$

  where $div(\sigma^\varepsilon) = (\sigma^\varepsilon_{ij,j})$ and $f^\varepsilon = (f_i^\varepsilon)_{1\le i\le 3}$ denotes the body forces.
- The stress tensor σ^ε is decomposed as follows

$$\begin{cases} \sigma^\varepsilon_{ij} = \tilde{\sigma}^\varepsilon_{ij} - p^\varepsilon \delta_{ij}, \\ \tilde{\sigma}^\varepsilon = \alpha^\varepsilon \dfrac{D(u^\varepsilon)}{|D(u^\varepsilon)|} + \mu \left|D(u^\varepsilon)\right|^{r-2} D(u^\varepsilon) \text{ if } D(u^\varepsilon) \neq 0, \\ |\tilde{\sigma}^\varepsilon| \le \alpha^\varepsilon \text{ if } D(u^\varepsilon) = 0. \end{cases} \tag{2}$$

  here $\alpha^\varepsilon \ge 0$ is the yield stress, $\mu > 0$ is the constant viscosity, u^ε is the velocity field, p^ε the pressure, δ_{ij} is the *Kronecker* symbol,

$1 < r \leq 2$ and $D(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$. For any tensor $D = (d_{ij})$, the notation $|D|$ represents the matrix norm: $|D| = \left(\sum_{i,j} d_{ij} d_{ij}\right)^{\frac{1}{2}}$.

- The incompressibility equation

$$div(u^\varepsilon) = 0 \text{ in } \Omega^\varepsilon. \tag{3}$$

Our boundary conditions is describe as

- At the surface $\Gamma_1 \cup \Gamma_L$ we assume that

$$u^\varepsilon = 0 \text{ in } \Gamma_1 \cup \Gamma_L. \tag{4}$$

- On ω, there is a no-flux condition across ω so that

$$u^\varepsilon \times n = 0, \tag{5}$$

- The tangential velocity on ω is unknown and satisfies *Tresca* boundary conditions:

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon \implies u_\tau^\varepsilon = 0 \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \implies \exists \lambda \geq 0, u_\tau^\varepsilon = -\lambda \sigma_\tau^\varepsilon \end{cases} \quad \text{in } \omega. \tag{6}$$

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Approximate solutions of nonlinear differential equations in their general form

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Abstract

In dynamics, the Van der Pol oscillator is a non-conservative oscillator with non-linear damping. It evolves in time according to the second-order differential equation.
In this work, we proved some general approximate analytical solutions to the general form of Van Der Pol equation

$$\ddot{u} + u + \epsilon(au^2 + b\dot{u}^2 - 1)\dot{u} = 0, \quad u(0) = A \text{ and } \dot{u}(0) = 0.$$

With a, b, ϵ real positive parameters and ϵ infinitely small.
We used in this study a different perturbation methods : Lindestedt-Poincaré method, averaging method, regular perturbation method and also renormalization group method.

Keywords

Perturbation theory, Van Der Pol equation, Lindestedt-Poincaré method, averaging method, renormalization group method.

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Titre : **Sur le théorème de Fuglede-Putnam**

Mots clé: opérateur borné, opérateur normal, Fuglede-Putnam.

Résumé

Dans ce présentation on s'intéresse au théorème classique et très important dans la théorie des opérateurs bornés, et non bornés avec toutes application à savoir le théorème de **Fuglede-Putnam**. Beaucoup d'auteurs travaillent sur ce théorème.

Le théorème de Fuglede fut établit en 1950 par B.Fuglede.

Théorème 1

Soient A et N deux opérateurs bornés sur un Hilbert tels que $AN = NA$, où N est normal. Alors,

$$AN^* = N^*A.$$

Puis en 1950 Putnam a fait la généralisation au cas de deux opérateurs normaux i.e;

Théoème 2

Soient A, M, N trois opérateurs bornés sur un Hilbert, avec N, M normaux et $MA = AM$. Alors,

$$AN^* = M^*A.$$

On remarque que les théorèmes originaux étaient démontrés pour deux opérateurs N, M pas nécerssairement bornés, dans un tel cas on remplace donc "$=$" par "$\subset$".

Il existe plusieurs versions de ce théorème pour les opérateurs " subnormaux, hyponormaux,p-hyponormaux, dominants, log-hyponormaux....etc" mais on va se contenter dans ce présentation aux quelques applications de ce thoérème pour les opérateurs bornés.

Application 1:

Sur la somme des deux opérateurs normaux.

Application 2:

Quelques conditions sur opérateurs non normaux qui impliquent normalité.

MULTIPLICITY RESULTS FOR STEKLOV PROBLEM INVOLVING THE $p(x)$-LAPLACIAN

SOURAYA FAREH AND KAMEL AKROUT

ABSTRACT. In the present paper, we are interested in the multiplicity of solutions for the Steklov problem involving the $p(x)$-Laplacian. By using variational methods and mountain pass Lemma combined with Ekeland variational principle, and for some hypothesis, we prove the existence of two nontrivial solutions.
KEYWORDS AND PHRASES. $p(x)$-Laplacian, Ekeland variational principle, Variational methods, Mountain pass theorem.

1. DEFINE THE PROBLEM

The aim of this paper, is to study the following Steklov boundary value problem
(1)

$$\begin{cases} -\operatorname{div}\left(a(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + |u|^{p(x)-2}\,u = f(x,u) + \lambda\,|u|^{\gamma(x)-2}\,u \text{ in } \Omega, \\ a(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial v} = b(x)\,|u|^{q(x)-2} \text{ on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$, is a bounded with Lipschitz boundary $\partial\Omega$, $\frac{\partial}{\partial v}$is the outer unit normal derivative, $(-\Delta)_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplacian, $p(x), \gamma(x)$ are continuous functions on $\overline{\Omega}$ such that $1 < p^- = \inf_{\overline{\Omega}} p(x) \leq p(x) \leq \sup_{\overline{\Omega}} p(x) = p^+$, we also denote γ^-, γ^+ for any $\gamma(x) \in C\left(\overline{\Omega}\right)$ and q^-, q^+ for any $q(x) \in C(\partial\Omega), p(x) \neq q(y)$ for any $x \in \overline{\Omega}, y \in \partial\Omega, a$ and b are continuous functions such that

$$a_1 \leq a(x) \leq a_2, \text{ and } b_1 \leq b(x) \leq b_2,$$

where a_1, a_2, b_1 and b_2 are positive constants, λ is a positive parameter, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a caratheodory function satisfying some conditions. More precisely, we assume the following hypothesis.
(a_0)There exist $C > 0, \alpha \in C(\overline{\Omega})$, such that for all $(x, u) \in \Omega \times \mathbb{R}$, we have

$$|f(x,u)| \leq C\left(1 + |u|^{\alpha(x)-1}\right),$$

and

$$1 < \alpha(x) < p^*(x).$$

(a_1) $f(x,u) = o\left(|u|^{p^+-1}\right)$ as $u \to 0$ and for all $x \in \Omega$.
(a_2) There exist $K > 0, \theta > p^+$ such that for all $x \in \Omega$, we have

$$0 < \theta F(x,u) \leq f(x,u)u, \ |u| \geq K,$$

where $F(x,t)=\int_0^t f(x,s)ds$.
(a_3) There exist $1<t<p^-$, such that

$$\liminf_{u\to 0}\frac{F(x,u)+\frac{\lambda}{\gamma(x)}|u|^{\gamma(x)}}{|u|^t}>0, \text{ and for all } x\in\Omega.$$

where

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)}, \text{ if } p(x)<N,\\ \\ \infty, \text{ if } p(x)\geq N.\end{cases}, p_*(x)=\begin{cases}\frac{(N-1)p(x)}{N-p(x)}, \text{ if } p(x)<N,\\ \\ \infty, \text{ if } p(x)\geq N.\end{cases},$$

we have our main result

Theorem 1.1. *Let $\alpha^->p^+,\theta>q^+$ and assume that hypothesis $(a_0)-(a_3)$ are satisfied, then, there exists $\lambda_0>0$ such that for every $\lambda\in(0,\lambda_0)$, problem (1) has at least two nontrivial weak solutions.*

For the poof of theorem 1.1, we will use the Mountain pass theorem combined with Ekeland variational principle, wich we mention below

Lemma 1.2. (*Mountain pass theorem*) (*see* [1]) *Let X be a Banach space, $\Psi\in C^1(X,\mathbb{R}), e\in X$ and $||e||>r$ for some $r>0$. Assume that*

$$\inf_{||u||=r}\Psi(u)>\Psi(0)\geq\Psi(e).$$

If Ψ satisfies the (PS) condition at level c, then, c is a critical value of Ψ, where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Psi(\gamma(t)),\ \Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0,\gamma(1)=e\}.$$

Lemma 1.3. (*Ekeland variational principle*)[2] *Let X be a banach space, $\Psi:X\to\mathbb{R}\cup\{+\infty\}$ a l.s.c function Câteaux-differentiable such that:*

$$-\infty<\inf\Psi<+\infty,$$

then, for every $\varepsilon>0$, for every $u\in X$ such that $\Psi(u)\leq\inf\Psi+\varepsilon$ every $\lambda>0$, there exists $v\in X$ such that:

$$\begin{aligned}\Psi(v) &\leq \Psi(u),\\ \|v-u\| &< \lambda,\\ \left\|\Psi'(v)\right\|^* &\leq \frac{\varepsilon}{\lambda}.\end{aligned}$$

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ITERATIVE COLLOCATION METHOD FOR SOLVING A CLASS OF NONLINEAR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this paper, an iterative collocation method based on the use of Lagrange polynomials is developed for the numerical solution of a class of nonlinear weakly singular Volterra integral equations. The error analysis of the proposed numerical method is studied theoretically. Numerical illustrations confirm our theoretical analysis.

1. INTRODUCTION

In this paper, we develop an approximation based on iterative collocation method to obtain numerical solutions of the following nonlinear weakly singular Volterra integral equations,

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, t \in I = [0,T], \tag{1.1}$$

where the functions g, k are sufficiently smooth and $p(t,s) = \frac{s^{\mu-1}}{t^{\mu}}$, $\mu > 1$.
Equations with this kind of kernel have a weak singularity at $t = 0$ and they are a particular case of the cordial equations, studied by G. Vainikko in [11], [12] and [13]. Actually, as shown in [11], if the core function of a cordial operator is $\phi(s) = s^{\mu-1}$, then its kernel is $k(t,s) = s^{\mu-1}t^{-\mu}$, which is the kind of kernel we are concerned with. Equations of this type are also the subject of the article [5].

The cordial integral operators have the interesting property that they are bounded but non-compact, which implies that some of the classical results for Volterra integral equations (for example, about existence and uniqueness of solution) are not applicable in this case. However an existence and uniqueness result in $C^m([0,T])$ was obtained in [11], provided that the core function satisfies $\phi(x) \in L^1([0,1])$, which is the case of our equation, when $\mu > 0$.

The application of polynomial and spline collocation methods to cordial equations was studied in [12] and [13], respectively, where sufficient conditions for convergence were obtained and error estimates were derived. Superconvergence results for collocation methods were obtained in [5].

Equations of this type arise from heat conduction problems. As it was shown in [4], they may result from boundary value problems for partial differential equations with mixed-type boundary conditions.

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In [5] the authors were concerned with the numerical solution of linear cordial equations. Here we propose a computational method for a nonlinear Volterra integral equation with a weakly singular kernel of the same type.

In [3] a similar approach was proposed for nonlinear Volterra integral equations with regular kernels (when $p(t,s) \equiv 1$). This case was also well studied in the literature. In particular, Babolian and his co-authors [2] have proposed a Chebyshev approximation. In [1] and [7] numerical algorithms based on the Adomian's method were developed. In [14] an approach was proposed, based on Taylor polynomial approximation, while the homotopy perturbation method was applied to the same equation in [6]. The authors of [8] have introduced a scheme based on the fixed point method. Finally, the Haar wavelet method and the Haar rationalized functions method were proposed in [9] and [10], respectively.

In Section 2 of the present work we describe a numerical scheme for the solution of equation 1.1. In Section 3 we analyze the convergence and obtain error estimates. Numerical examples that illustrate the performance of the method are presented in Section 4 and the paper finishes with conclusions in Section 5.

2. Description of the collocation method

Let Π_N be a uniform partition of the interval $I = [0,T]$ defined by $t_n = nh, \quad n = 0, ..., N-1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 < c_1 < ...... < c_m \leq 1$ and the collocation points be $t_{n,j} = t_n + c_j h$, $j = 1, ..., m, n = 0, ..., N-1$. Define the subintervals $\sigma_n = [t_n, t_{n+1}[$, and $\sigma_{N-1} = [t_{N-1}, t_N]$.
Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree $m-1$ as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1}, n = 0, .., N-1\}.$$

This is the space of piecewise polynomials of degree at most m. Its dimension is Nm. It holds for any $y \in C^m([0,T])$ that

$$y(t_n + \tau h) = \sum_{l=1}^{m} \lambda_l(\tau) y(t_{n,j}) + \epsilon_n(\tau),\ \epsilon_n(\tau) = h^m \frac{y^{(m)}(\zeta_n(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j), \quad (2.1)$$

where $\tau \in [0,1]$ and $\lambda_j(\tau) = \prod\limits_{l \neq j}^{m} \frac{\tau - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters $c_j, j = 1, ..., m$. Let $\Gamma_m = \| \sum\limits_{j=1}^{m} |\lambda_j| \|$.
Inserting (2.1) for the function $s \longmapsto K(t, s, x(s))$ into (1.1), we obtain for each $j = 1, ..., m, , n = 0, ..., N-1$

$$x(t_{nj}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, x(t_{nl})) \lambda_l(\tau) d\tau \right) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, x(t_{il})) \lambda_l(\tau) d\tau \right) + o(h^m). \quad (2.2)$$

It holds for any $u \in S_{m-1}^{-1}(I, \Pi_N)$ that

$$u(t_n + \tau h) = \sum_{l=1}^{m} \lambda_l(\tau) u(t_{n,l}), \tau \in [0, 1]. \tag{2.3}$$

Now, we approximate the exact solution x by $u \in S_{m-1}^{-1}(I, \Pi_N)$ such that $u(t_{n,j})$ satisfy the following nonlinear system,

$$u(t_{n,j}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u(t_{nl})) \lambda_l(\tau) d\tau \right) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u(t_{il})) \lambda_l(\tau) d\tau \right). \tag{2.4}$$

for $j = 1, ..., m$, $n = 0, ..., N-1$.
Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_{m-1}^{-1}(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1.1) such that

$$u^q(t_n + \tau h) = \sum_{j=1}^{m} \lambda_j(\tau) u^q(t_{n,j}), \tau \in [0, 1] \tag{2.5}$$

where the coefficients $u^q(t_{n,j})$ are given by the following formula:

$$u^q(t_{n,j}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u^{q-1}(t_{nl})) \lambda_l(\tau) d\tau \right) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u^q(t_{il})) \lambda_l(\tau) d\tau \right). \tag{2.6}$$

such that the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).
The above formula is explicit and the approximate solution u^q is obtained without solving any algebraic system.
In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (1.1).

3. Convergence analysis

In this section, we assume that the function k satisfies the Lipschitz condition with respect to the third variable: there exists $L \geq 0$ such that

$$|k(t, s, y_1) - k(t, s, y_2)| \leq L|y_1 - y_2|, \tag{3.1}$$

for all $t, s \in I$, where L is independent of t and s.
The following result gives the existence and the uniqueness of a solution for the nonlinear system (2.4).

Lemma 3.1. *If $\frac{L\Gamma_m}{\mu} < 1$, then the nonlinear system (2.4) has a unique solution $u \in S_{m-1}^{-1}(I, \Pi_N)$. Moreover, the function u is bounded.*

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0=[t_0,t_1]$, the nonlinear system (2.4) becomes

$$u(t_{0,j})=g(t_{0,j})+\sum_{l=1}^{m}\left(\int_0^{c_j}\frac{(\tau)^{\mu-1}}{(c_j)^{\mu}}k(t_0+c_jh,t_0+c_lh,u(t_{0l}))\lambda_l(\tau)d\tau.\right)$$

We consider the operator Ψ defined by

$$\Psi:\mathbb{R}^m\longrightarrow\mathbb{R}^m$$
$$x=(x_1,...,x_m)\longmapsto\Psi(x)=(\Psi_1(x),...,\Psi_m(x)),$$

such that for $j=1,...,m$, we have

$$\Psi_j(x)=g(t_{0,j})+\sum_{l=1}^{m}\left(\int_0^{c_j}\frac{(\tau)^{\mu-1}}{(c_j)^{\mu}}k(t_0+c_jh,t_0+c_lh,x_l)\lambda_l(\tau)d\tau\right).$$

Hence, for all $x,y\in\mathbb{R}^m$, we have

$$\|\Psi(x)-\Psi(y)\|\leq\frac{L\Gamma_m}{\mu}\|x-y\|,$$

Since $\frac{L\Gamma_m}{\mu}<1$, then by Banach fixed point theorem, the nonlinear system (2.4) has a unique solution u on the interval σ_0.

(ii) Suppose that u exists and is unique on the intervals $\sigma_i, i=0,...,n-1$ for $n\geq 1$, we show now that u exists and is unique on the interval σ_n.
On the interval σ_n, the nonlinear system (2.4) becomes

$$u(t_{n,j})=G(t_{n,j})+\sum_{l=1}^{m}\left(\int_0^{c_j}\frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}}k(t_n+c_jh,t_n+c_lh,u(t_{nl}))\lambda_l(\tau)d\tau\right) \tag{3.2}$$

where,
$G(t_{n,j})=g(t_{n,j})+\sum\limits_{i=0}^{n-1}\sum\limits_{l=1}^{m}\left(\int_0^1\frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}}k(t_n+c_jh,t_i+c_lh,u(t_{il}))\lambda_l(\tau)d\tau\right)$.
We consider the operator Ψ defined by:

$$\Psi:\mathbb{R}^m\longrightarrow\mathbb{R}^m$$
$$x=(x_1,...,x_m)\longmapsto\Psi(x)=(\Psi_1(x),...,\Psi_m(x)),$$

such that for $j=1,...,m$, we have

$$\Psi_j(x)=G(t_{n,j})+\sum_{l=1}^{m}\left(\int_0^{c_j}\frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}}k(t_n+c_jh,t_n+c_lh,x_l)\lambda_l(\tau)d\tau\right).$$

Hence, for all $x,y\in\mathbb{R}^m$, we have

$$\|\Psi(x)-\Psi(y)\|\leq\frac{L\Gamma_m}{\mu}\|x-y\|.$$

Since $\frac{L\Gamma_m}{\mu}<1$, then by Banach fixed point theorem, the nonlinear system (3.2) has a unique solution u on the interval σ_n.

□

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 3.2. *Let f, k be m times continuously differentiable on their respective domains. If $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then the collocation solution u converges to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$\|e\| \leq Ch^m,$$

where C is a finite constant independent of h.

Proof. From (2.4) and (2.2), using (3.1), we obtain

$$|e(t_{nj})| \leq \alpha h^m + \frac{L}{\mu} \sum_{l=1}^{m} |\lambda_l||e(t_{nl})| + \frac{L}{\mu n^\mu} \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left((i+1)^\mu - i^\mu\right) |e(t_{il})| \tag{3.3}$$

for all $n = 0, ..., N-1$ and $j = 1, ..., m$, where α is a positive number.
We consider the sequence $e_n = \max\{|e(t_{n,l})|, l = 1, ..., m\}$ for $n = 0, ..., N-1$.
Then, from (3.3), e_n satisfies for $n = 0, ..., N-1$,

$$e_n \leq \alpha h^m + \frac{L\Gamma_m}{\mu} e_n + \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i,$$

which implies that,

$$e_n \leq \frac{\alpha}{1 - \frac{L\Gamma_m}{\mu}} h^m + \frac{L\Gamma_m}{(1 - \frac{L\Gamma_m}{\mu})\mu n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i.$$

Let $C_1 = \frac{\alpha}{1 - \frac{L\Gamma_m}{\mu}}$ and $C_2 = \frac{L\Gamma_m}{\mu(1 - \frac{L\Gamma_m}{\mu})}$, it follows that

$$e_n \leq C_1 h^m + \frac{C_2}{n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i.$$

Hence, for $\xi = \max\{e_n, n = 0, ..., N-1\}$, we deduce that

$$\xi \leq C_1 h^m + C_2 \xi.$$

Since $C_2 < 1$, we obtain

$$\xi \leq \frac{C_1}{1 - C_2} h^m.$$

Which implies, from (2.1) and (2.3), that there exists $C > 0$ such that

$$\begin{aligned}\|e\| &\leq \Gamma_m \xi + h^m \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1 - c_j) \\ &\leq \Gamma_m \frac{C_1}{1 - C_2} h^m + h^m \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1 - c_j).\end{aligned}$$

Thus, the proof is completed by setting $C = \Gamma_m \frac{C_1}{1 - C_2} + \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1 - c_j)$. □

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 3.3. *Consider the iterative collocation solution $u^q, q \geq 1$ defined by (2.5) and (2.6). If $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then for any initial condition $u^0(t_{n,j}) \in J$ (bounded interval), the iterative collocation solution $u^q, q \geq 1$ converges to the exact solution x. Moreover, the following error estimate holds*

$$\|u^q - x\| \leq d\rho^q + Ch^m$$

where d, C are finite constants independent of h and $\rho < 1$.

Proof. We define the error e^q and ξ^q by $e^q(t) = u^q(t) - x(t)$ and $\xi^q(t) = u^q(t) - u(t)$, where u is defined by lemma 3.1. It follows that

$$e^q = \xi^q + u - x. \tag{3.4}$$

We have, from (2.4) and (2.6), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$|\xi^q(t_{n,j})| \leq \frac{L\Gamma_m}{n^\mu \mu} \sum_{i=0}^{n-1} \sum_{l=1}^{m} [(i+1)^\mu - i^\mu] |\xi^q(t_{il})| + \frac{L\Gamma_m}{\mu} \sum_{l=1}^{m} \left|\xi^{q-1}(t_{n,l})\right|. \tag{3.5}$$

Now, for each fixed $q \geq 1$, we consider the sequence $\xi_n^q = \max\{|\xi^q(t_{n,l})|, l = 1....m\}$ for $n = 0, ..., N-1$, it follows from (3.5) that,

$$\xi_n^q \leq \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} [(i+1)^\mu - i^\mu] \xi_i^q + \frac{L\Gamma_m}{\mu} \xi_n^{q-1}.$$

We consider the sequence $\eta^q = \max\{\xi_n^q, n = 0, ..., N-1\}$ for $q \geq 1$.
Then, η^q satisfies,

$$\begin{aligned}\eta^q &\leq \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} [(i+1)^\mu - i^\mu] \eta^q + \frac{L\Gamma_m}{\mu} \eta^{q-1} \\ &\leq \frac{L\Gamma_m}{\mu} \eta^q + \frac{L\Gamma_m}{\mu} \eta^{q-1}.\end{aligned}$$

Hence,

$$\eta^q \leq \rho \eta^{q-1}, \tag{3.6}$$

where $\rho = \dfrac{\frac{L\Gamma_m}{\mu}}{1 - \frac{L\Gamma_m}{\mu}}$, since $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then $\rho < 1$.
Which implies, from (3.6), that for all $q \geq 1$, that

$$\eta^q \leq \rho \eta^{q-1} \leq \rho^2 \eta^{q-2} \leq ... \leq \rho^q \eta^0 \leq \rho^q \|\xi^0\|. \tag{3.7}$$

Since, $u^0(t_{n,j}) \in J$, the function u^0 is bounded.
Hence, there exists $M > 0$ such that

$$\|\xi^0\| = \|u^0 - u\| \leq \|u^0 - x\| + \|u - x\| \leq M. \tag{3.8}$$

From (3.7) and (3.8), we conclude that

$$\|\xi^q\| \leq \Gamma_m \eta^q \leq \underbrace{\Gamma_m M}_{d} \rho^q.$$

On the other hand, from Theorem (3.2), we have $\|u - x\| \leq Ch^m$ and therefore by (3.4) we obtain

$$\|e^q\| \leq \|\xi^q\| + \|u - x\| \leq d\rho^q + Ch^m.$$

Thus, the proof is completed. □

4. Numerical Examples

To illustrate the theoretical results obtained in the previous section, we present the following examples with $T = 1$. All the exact solutions x are already known. In each example, we calculate the error between x and the iterative collocation solution u^q for $N = 10, 20$ and $m = 2, 3, 5$ at $t = 0, 0.1, ..., 1$. In all the examples, we choose, $q = 5$, $u^0(t_{nj}) = 1$, and we use the collocation parameters $c_j = \frac{j}{m+1}, j = 1, ..., m$. Since the condition $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$ is essential to guarantee the convergence of the numerical method, we checked that it is satisfied in all the numerical examples, for $m = 3$ and $m = 4$. For $m = 5$, since it is very difficult to obtain the norm Γ_m, we have checked that this condition is satisfied at a finite set of uniformly distributed points of the interval $[0, 1]$.

Example 4.1. *Consider the following integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \; t \in [0,1].$$

with $k(t,s,z) = \frac{st\exp(z)}{11(1+\exp(z))}$, $\mu = 2$ *and* $g(t)$ *is chosen such that the exact solution of this equation is* $x(t) = \ln(1 + t^2)$. *The absolute errors are presented in Table 1.*

Table 1. Absolute errors for Example 4.1

	$N = 10$	$N = 10$	$N = 10$	$N = 20$	$N = 20$	$N = 20$
t	$m = 2$	$m = 3$	$m = 5$	$m = 2$	$m = 3$	$m = 5$
0	2.21×10^{-3}	6.97×10^{-6}	1.63×10^{-8}	5.55×10^{-4}	4.37×10^{-7}	3.64×10^{-9}
0.1	2.10×10^{-3}	2.41×10^{-5}	3.45×10^{-8}	5.33×10^{-4}	2.65×10^{-6}	1.68×10^{-11}
0.2	1.89×10^{-3}	3.70×10^{-5}	5.22×10^{-8}	4.83×10^{-4}	4.38×10^{-6}	8.72×10^{-9}
0.3	1.60×10^{-3}	4.40×10^{-5}	5.04×10^{-8}	4.13×10^{-4}	5.39×10^{-6}	3.44×10^{-9}
0.4	1.28×10^{-3}	4.53×10^{-5}	3.11×10^{-8}	3.33×10^{-4}	5.69×10^{-6}	1.18×10^{-9}
0.5	9.67×10^{-4}	4.25×10^{-5}	2.02×10^{-8}	2.54×10^{-4}	5.41×10^{-6}	8.58×10^{-9}
0.6	6.82×10^{-4}	3.71×10^{-5}	1.02×10^{-8}	1.81×10^{-4}	4.78×10^{-6}	2.24×10^{-9}
0.7	4.39×10^{-4}	3.07×10^{-5}	4.35×10^{-9}	1.19×10^{-4}	3.99×10^{-6}	4.42×10^{-10}
0.8	2.42×10^{-4}	2.44×10^{-5}	3.53×10^{-9}	6.78×10^{-5}	3.19×10^{-6}	3.26×10^{-9}
0.9	8.80×10^{-5}	1.87×10^{-5}	5.27×10^{-9}	2.76×10^{-5}	2.46×10^{-6}	2.47×10^{-9}
1	4.42×10^{-5}	1.74×10^{-5}	6.05×10^{-8}	6.09×10^{-6}	2.06×10^{-6}	3.84×10^{-9}

Example 4.2. *Consider the following integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \; t \in [0,1].$$

with $k(t,s,z) = \frac{ts}{10(1+z^2)}$, $\mu = 2$ *and* $g(t)$ *is chosen so that the exact solution of this equation is* $x(t) = \frac{3t}{t+1}$. *The absolute errors are presented in Table 2.*

Example 4.3. *Consider the following integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \; t \in [0,1].$$

with $k(t,s,z) = \frac{t\cos(s+z)}{50}$, $\mu = 1.03$ *and* $g(t)$ *is chosen such that the exact solution of this equation is* $x(t) = \frac{t}{10}$. *The absolute errors are presented in Table 3.*

Table 2. Absolute errors for Example 4.2

	$N=10$	$N=10$	$N=10$	$N=20$	$N=20$	$N=20$
t	$m=2$	$m=3$	$m=5$	$m=2$	$m=3$	$m=5$
0	8.06×10^{-3}	3.23×10^{-4}	4.81×10^{-7}	2.11×10^{-3}	4.34×10^{-5}	1.80×10^{-8}
0.1	6.12×10^{-3}	2.24×10^{-4}	2.79×10^{-7}	1.59×10^{-3}	2.99×10^{-5}	1.00×10^{-8}
0.2	4.75×10^{-3}	1.60×10^{-4}	1.72×10^{-7}	1.23×10^{-3}	2.12×10^{-5}	6.43×10^{-8}
0.3	3.76×10^{-3}	1.17×10^{-4}	9.39×10^{-8}	9.76×10^{-4}	1.55×10^{-5}	7.92×10^{-9}
0.4	3.03×10^{-3}	8.79×10^{-5}	6.71×10^{-8}	7.84×10^{-4}	1.15×10^{-5}	1.21×10^{-8}
0.5	2.47×10^{-3}	6.71×10^{-5}	4.46×10^{-8}	6.39×10^{-4}	8.81×10^{-6}	6.66×10^{-9}
0.6	2.04×10^{-3}	5.21×10^{-5}	3.80×10^{-8}	5.27×10^{-4}	6.82×10^{-6}	1.30×10^{-8}
0.7	1.71×10^{-3}	4.11×10^{-5}	2.44×10^{-8}	4.40×10^{-4}	5.37×10^{-6}	1.34×10^{-8}
0.8	1.44×10^{-3}	3.29×10^{-5}	1.92×10^{-8}	3.72×10^{-4}	4.29×10^{-6}	5.22×10^{-9}
0.9	1.23×10^{-3}	2.66×10^{-5}	2.68×10^{-8}	3.16×10^{-4}	3.46×10^{-6}	4.10×10^{-9}
1	1.17×10^{-3}	2.52×10^{-5}	4.31×10^{-7}	2.85×10^{-4}	3.04×10^{-6}	5.50×10^{-8}

Table 3. Absolute errors for Example 4.3

	$N=10$	$N=10$	$N=10$	$N=20$	$N=20$	$N=20$
t	$m=2$	$m=3$	$m=5$	$m=2$	$m=3$	$m=5$
0	4.36×10^{-7}	2.38×10^{-10}	3.60×10^{-11}	5.46×10^{-8}	9.00×10^{-12}	3.00×10^{-12}
0.1	1.08×10^{-6}	7.00×10^{-10}	1.90×10^{-10}	2.17×10^{-7}	3.00×10^{-11}	8.00×10^{-11}
0.2	1.71×10^{-6}	1.17×10^{-9}	1.30×10^{-10}	3.76×10^{-7}	1.00×10^{-10}	2.00×10^{-11}
0.3	2.32×10^{-6}	1.60×10^{-9}	7.00×10^{-11}	5.31×10^{-7}	1.30×10^{-10}	1.40×10^{-10}
0.4	2.91×10^{-6}	2.03×10^{-9}	1.30×10^{-10}	6.79×10^{-7}	8.00×10^{-11}	1.30×10^{-10}
0.5	3.45×10^{-6}	2.40×10^{-9}	1.40×10^{-10}	8.19×10^{-7}	1.10×10^{-10}	1.30×10^{-10}
0.6	3.95×10^{-6}	2.86×10^{-9}	6.00×10^{-11}	9.48×10^{-7}	2.20×10^{-10}	2.30×10^{-10}
0.7	4.40×10^{-6}	3.19×10^{-9}	8.00×10^{-11}	1.06×10^{-6}	2.40×10^{-10}	3.00×10^{-11}
0.8	4.80×10^{-6}	3.40×10^{-9}	2.30×10^{-10}	1.16×10^{-6}	1.30×10^{-10}	3.00×10^{-11}
0.9	5.13×10^{-6}	3.76×10^{-9}	3.00×10^{-11}	1.25×10^{-6}	2.70×10^{-10}	2.80×10^{-10}
1	4.99×10^{-6}	3.76×10^{-9}	9.70×10^{-10}	1.27×10^{-6}	2.70×10^{-10}	4.30×10^{-9}

5. Conclusion

In this paper, we have used an iterative collocation method based on the Lagrange polynomials for solving a class of nonlinear weakly singular Volterra integral equations (1.1) in the spline space $S_{m-1}^{(-1)}(\Pi_N)$. The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of approximate solution are determined by using iterative formulas without solving any system of algebraic equations. The numerical examples confirm that the method is convergent, very effective and convenient.

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A COMBINATION OF THE TWO DOMAIN DECOMPOSITION METHODS FOR THE NUMERICAL RESOLUTION OF AN EVOLUTION PROBLEM.

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Abstract. In this paper we present a combination of the two domain decomposition methods (overlapping and nonoverlapping methods) with Dirichlet boundary conditions on the interfaces for the numerical resolution of an parabolic variational equation with second order boundary value problem using the semi-implicit-time scheme combined with a finite element spatial approximation.

1. Presentation of the problem

We consider the following parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f, \quad \text{in } \Sigma, \\ u = 0, \quad \text{in } \Gamma \times [0,T], \\ u(.,0) = u_0, \quad \text{in } \Omega. \end{cases} , \tag{1.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.

The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative; f is a regular function.

Using the Green Formula, this problem can be transformed into the following continuous parabolic variational equation: find $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ solution to

$$\begin{cases} (u_t, v)_\Omega \quad + a(u,v) =, (f,v)_\Omega \;, \; v \in H_0^1(\Omega) \\ u(.,0) = u_0. \end{cases} \tag{1.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v.dx + \int_\Omega \alpha uv.dx \; .$$

We first discretize the problem (2.1) with respect to time by using the semi–implicit scheme. Therefore, we search a sequence of elements $u^k \in H_0^1(\Omega)$ which approaches $u^i(t_k)$, $t_k = k\Delta t$, with initial data $u^{i,0} = u_0^i$.

2000 *Mathematics Subject Classification.* Primary 65A38, 15A15; secondary 35A15, 35A18.

Key words and phrases. parabolic PDE; finite element; doverlapping and nonoverlapping domain decomposition methods.

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Thus, we have for $k = 1, ..., n$ the following sequence of elliptic variational equation

$$\begin{cases} b\left(u^k, v\right) = \left(f^k + \lambda u^{k-1}, v\right) = \left(F\left(u^{k-1}\right), v\right), \\ u^0(x) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

such that

$$\begin{cases} b\left(u^k, v\right) = \lambda\left(u^k, v\right) + a\left(u^k, v\right),\ u^k \in\ H_0^1(\Omega) \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{1}{k} = \dfrac{T}{n},\ k = 1, ..., n. \end{cases} . \tag{1.4}$$

2. A combination of overlapping and non overlapping domain decomposition for the numerical resolution

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12}, \qquad \partial\Omega_i \cap \Omega_j = \Gamma_i, \qquad i \neq j \text{ and } i, j = 1, 2.$$

We define the continuous counterparts of the continuous Schwarz sequences defined in (1.3), respectively by $u_1^{k,m+1} \in H_0^1(\Omega)$ and $u_2^{k,m+1} \in H_0^1(\Omega)$, $m = 0, 1, 2, ...$ such that

$$\begin{cases} b\left(u_1^{k,m+1}, v\right) = \left(F\left(u_1^{k-1,m+1}\right), v\right)_{\Omega_1} \\ u_1^{k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1 \\ \dfrac{\partial u_1^{k,m+1}}{\partial\eta_1} + \alpha_1 u_1^{k,m+1} = \dfrac{\partial u_2^{k,m}}{\partial\eta_1} + \alpha_1 u_2^{k,m},\ \text{on } \Gamma_1 \end{cases} \tag{2.1}$$

$$\begin{cases} b\left(u_2^{k,m+1}, v\right) = \left(F\left(u_2^{k-1,m+1}\right), v\right)_{\Omega_2}, \quad m = 0, 1, 2, .. \\ u_2^{k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega = \partial\Omega_2 - \Gamma_2 \\ \dfrac{\partial u_2^{k,m+1}}{\partial\eta_2} + \alpha_2 u_2^{n+1,m+1} = \dfrac{\partial u_1^{k,m}}{\partial\eta_2} + \alpha_2 u_1^{k,m},\ \text{on } \Gamma_2. \end{cases} \tag{2.2}$$

Since it is numerically easier to compare the subdomain solutions on the interfaces Γ_1 and Γ_2 rather than on the overlap Ω_{12}, thus we need to introduce two auxiliary problems defined on nonoverlapping subdomains of Ω. This idea allows us to obtain the a posteriori error estimate by following the steps of Otto and Lube [1]. We get these auxiliary problems by coupling each one of the problems (2.1) and (2.2) with another problem in a nonoverlapping way over Ω. These auxiliary problems are needed for the analysis and not for the computation,to get the estimate.

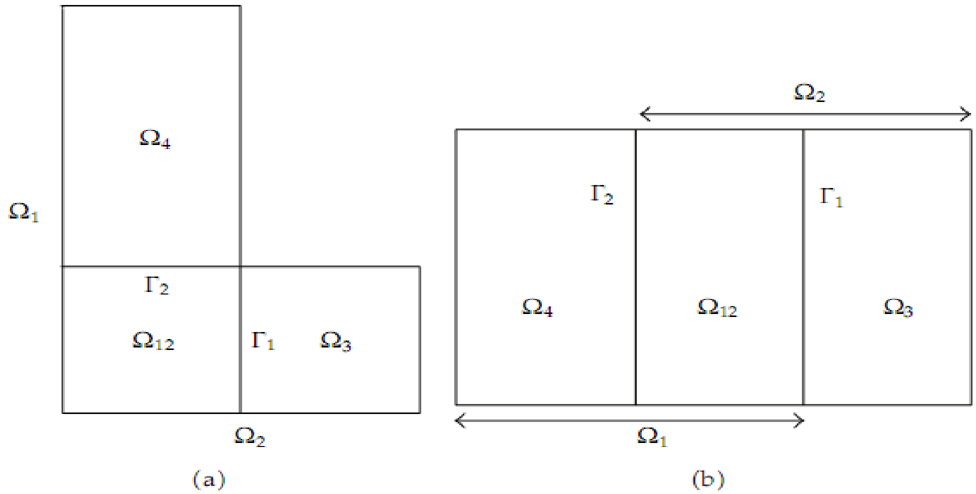

Figure 1: Two examples of domain decompositions (Schur and Schwarz).

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains (see figure 1) : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing \quad \Omega = \Omega_2 \cup \Omega_4, \text{ with } \Omega_2 \cap \Omega_4 = \varnothing.$$

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ON THE CONTROL OF A NONLINEAR SYSTEM OF VISCOELASTIC EQUATIONS

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Abstract. In this work we establish a general decay rate for a nonlinear viscoelastic wave equation with boundary dissipation where the relaxation function satisfies $g'(t) \leq -\xi(t) g^p(t)$, $t \geq 0$, $1 \leq p \leq \frac{3}{2}$. This work generalizes and improves earlier results in the literature.

1. Introduction

It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. From the mathematical point of view, their memory effects are modeled by integrodifferential equations. Hence, questions related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years.

In the present work, we are concerned with

$$\begin{cases} u_{tt} - k_0 \triangle u(t) + \int_0^t g(t-s)\, div\,(a(x) \nabla u(s))\, ds + b(x) u_t = |u|^{\gamma-2} u, & \text{in } \Omega \times (0,\infty) \\ k_0 \dfrac{\partial u}{\partial \nu} - \int_0^t g(t-s)(a(x)\nabla u(s)).\nu ds + h(u_t) = 0, & \text{on } \Gamma_1 \times (0,\infty) \\ u = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, & x \in \Omega. \end{cases} \tag{1.1}$$

Where $k_0 > 0$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary, $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint with meas$(\Gamma_0) > 0$, and ν is the unit outward normal to Γ. $b : \Omega \to \mathbb{R}^+$ is a function, and

$$\begin{aligned} 2 &< \gamma \leq \frac{2n}{n-2}, \quad n \geq 3, \\ \gamma &> 2, \quad \text{if } n = 1,\ 2. \end{aligned} \tag{1.2}$$

Our aim in this work is to obtain a more general and explicit energy decay formula, from which the usual exponential and polynomial decay rates are only special cases of our result. In fact, our decay formulae extend and improve some results in the literature.

2. Preliminaries

In this section we prepare some material needed in the proof of our result. According to (1.2) we have the

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function such that

$$g(0) > 0, \quad k_0 - \int_0^\infty g(s)\, ds = l > 0. \tag{2.1}$$

Key words and phrases. Viscoelastic ; General decay ; Relaxation function ; Dissipation ; Wave equation.

$(A2)$ There exists a nonincreasing differentiable function $\xi:\mathbb{R}_+\rightarrow\mathbb{R}_+$, with $\xi(0)>0$, satisfying

$$g'(t)\leq-\xi(t)\,g^p(t),\quad\forall t\geq0,\ 1\leq p<\frac{3}{2}. \tag{2.2}$$

$(A3)$ $h:\mathbb{R}\rightarrow\mathbb{R}$ is a nondecreasing function with

$$h(s)\,s\geq\alpha\,|s|^2,\quad\forall s\in\mathbb{R}, \tag{2.3}$$

$$|h(s)|\leq\beta\,|s|,\quad\forall s\in\mathbb{R}. \tag{2.4}$$

$(A4)$ $a:\Omega\rightarrow\mathbb{R}$ is a nonnegative functions and $a\in C^1\left(\bar{\Omega}\right)$ such that

$$\begin{array}{rcl} a(x) & \geq & a_0>0, \\ |\nabla a(x)|^2 & \leq & a_1^2\,|a(x)|, \end{array} \tag{2.5}$$

for some positive constant a_1.

By using the Galerkin method and procedure similar to that of [10], we have the following local existence result for problem (1.1).

Theorem 2.1. *Let hypotheses* $(A1)$-$(A4)$ *hold and* (1.2) *hold and assume that* $u_0\in H^1_{\Gamma_0}\cap H^2(\Omega)$, $u_1\in H^1_{\Gamma_0}$. *Then there exists a strong solution* u *of* (1.1) *satisfying*

$$\begin{array}{rcl} u & \in & L^\infty\left([0,T);H^1_{\Gamma_0}\cap H^2(\Omega)\right) \\ u_t & \in & L^\infty\left([0,T);H^1_{\Gamma_0}\right) \\ u_{tt} & \in & L^\infty\left([0,T);L^2(\Omega)\right), \end{array}$$

for some $T>0$.

We introduce the following functionals

$$\begin{array}{rcl} J(t) & = & \frac{1}{2}\left(k_0-a(x)\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{\gamma}\|u\|_\gamma^\gamma \\ E(t) & = & J(u(t))+\frac{1}{2}\|u_t\|_2^2,\quad\text{for } t\in[0,T) \\ I(t) & = & I(u(t))=\left(k_0-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)-\|u\|_\gamma^\gamma, \end{array} \tag{2.6}$$

where

$$(g\circ v)(t)=\int_0^t g(t-s)\,\|v(t)-v(s)\|_2^2\,ds, \tag{2.7}$$

and $E(t)$ is the energy functional.

A direct differentiation, using (1.1), leads to

$$E'(t)=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}\int_\Omega a(x)\,g(t)\,|\nabla u(t)|^2\,dx-\int_\Omega b(x)\,|u_t(t)|^2\,dx\leq0. \tag{2.8}$$

We start with the following crucial lemma which will be used in the proof of our result.

The next lemma and corollary are crucial for the proof of our main result.

3. Global existence

We state and prove the global existence result.for all $t \in [0, T_m].T_m$ is extended to T.

Theorem 3.1. *Suppose that* $(A1), (A2)$. *If* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. *Then the solution is global and bounded.since* $I(t)$ *and* $(g \circ \nabla u)(t)$ *are positive. Therefore*

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq CE(0),$$

where c *is a positive constant, which depends only on* γ *and* l.

4. Decay of solutions

In this section we state and prove the main result of our work. First, we define some functionals. Let

$$\mathcal{F}(t) = E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \Psi(t), \tag{4.1}$$

where

$$\Phi(t) = \int_\Omega u.u_t \, dx, \tag{4.2}$$

$$\Psi(t) = \int_\Omega a(x) u_t \int_0^t g(t-s)(u(s) - u(t)) \, ds \, dx, \tag{4.3}$$

and ε_1, ε_2 are some positive constants to be specified later.

Lemma 4.1. *There exist two positive constants* β_1 *and* β_2 *such that the relation*

$$\beta_1 E(t) \leq \mathcal{F}(t) \leq \beta_2 E(t). \tag{4.4}$$

We are ready to state and prove our main result.

Theorem 4.2. *Let* $(u_0, u_1) \in \left(H^1_{\Gamma_0} \times L^2(\Omega)\right)$ *be given . Assume that* $(A1)$*-*$(A4)$ *are satisfied. Then, for any* $t_0 > 0$, *there exist two positive constants* K, *and* λ *such that the solution of* (1.1) *satisfies, for all* $t \geq t_0$,

$$E(t) \leq K e^{-\lambda \int_{t_0}^t \xi(s) ds}, \qquad \text{if } p = 1. \tag{4.5}$$

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s) \, ds} \right]^{\frac{1}{2p-2}}, \quad p > 1. \tag{4.6}$$

Moreover, if

$$\int_0^{+\infty} \left[\frac{1}{t\xi^{2p-1}(t) + 1} \right]^{\frac{1}{2p-2}} dt < +\infty, \quad 1 < p < \frac{3}{2}, \tag{4.7}$$

then

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^t \xi^p(s) \, ds} \right]^{\frac{1}{p-1}}, \qquad p > 1. \tag{4.8}$$

Simple calculations show that (4.6) *and* (4.7) *yield*

$$\int_{t_0}^{+\infty} E(t) \, dt < +\infty.$$

Proof. The following examples illustrate our result given by Theorem 4.2. □

Example 1. *Let* $g(t) = \frac{a}{(1+t)^{\nu}}$, $\nu > 2$, *where* $a > 0$ *is a constant so that* $\int_0^{+\infty} g(t)\,dt < 1$. *We have*

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^{\nu}}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \quad p = \frac{\nu+1}{\nu} < \frac{3}{2}, \quad b > 0.$$

Therefore (4.7), *with* $\xi(t) = b$, *yields* $\int_0^{+\infty} \left(\frac{1}{b^{2p-1}t+1}\right)^{\frac{1}{2p-2}} dt < \infty$ *and hence by* (4.8) *we get*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p-1}}} = \frac{C}{(1+t)^{\nu}},$$

which is the optimal decay obtained in [7].

Example 2. *Let* $g(t) = ae^{-(1+t)^{\nu}}$, $0 < \nu \leq 1$ *where* $0 < a < 1$ *is chosen so that* $\int_0^{+\infty} g(t)\,dt < 1$. *Then*

$$g'(t) = -a\nu(1+t)^{\nu-1} e^{-(1+t)^{\nu}} = -\xi(t)\,g(t)$$

where $\xi(t) = \nu(1+t)^{\nu-1}$ *which is a decreasing function and* $\xi(0) > 0$. *Therefore we can use* (4.5) *to deduce*

$$E(t) \leq Ce^{-\lambda(1+t)^{\nu}}.$$

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Multigrid Methods for the Solution of Variational Inequalities.

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1-Summary

This work concerns the study of a class of variational inequalities, in the sense that the second member depends on the solution, applying the methods multigrids in two domain .

Find $u \in K_g$ solution of

$$\begin{cases} \Delta u \geq f(u) \quad in \quad \Omega \\ u \leq \psi; \quad \psi \geq 0 \\ u = g; \quad on \quad \partial\Omega \end{cases}$$

where $f \in L^{\infty}(\Omega)$ and

$$K_g = \left\{ v \in H^1(\Omega) : v = g \quad on \quad \partial\Omega, v \leq \psi \quad in \quad \Omega \right\}$$

Multigrid methods consist in successively using grids (or meshes) of different sizes, so as to obtain a detailed solution in the high frequencies, while ensuring rapid relaxation of the basic frequencies. Multigrid methods have been studied for linear elliptical problems. For our part, we are interested in the finite difference approximation, in using multigrid algorithms, for variational inequalities nonlinear, as the second member depends on the solution.

2-Keywords

Multigrid method, variational inequality, finite element, iterative method, HJB equation.

3-Abstract

We study the numerical solution of second member problems depending on the solution by a multigrid method. We prove the uniform convergence of the multigrid algorithm using the elementary subdifferential calculus and ideas from the convergence theory of nonlinear multigrid methods.

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Time-consistent investment and consumption strategies for Markovian Merton model under a general discount function

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Abstract

In this presentation, we investigate equilibrium consumption-investment for Merton's portfolio problem with non-exponential discount function and general utility function. We consider that the market coefficients according to a finite state Markov chain. The non-exponential discount in the objective function is the reason for the time-inconsistency in our paper. Since this problem is time-inconsistent we treat it by placing within a game theoretic framework and look for subgame perfect Nash equilibrium strategies. Using a variational technical approach, we derive the necessary and sufficient equilibrium condition, also we provide a verification theorem for an open-loop equilibrium consumption-investment strategy. An closed loop representation of the equilibrium is derived within special forms of the utility function (logarithme and power).

Keys words: Investment-Consumption Problem, Merton Portfolio Problem, Equilibrium Strategies, Time Inconsistency, Non-Exponential Discounting.

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Exponential decay for a strain gradient porous thermoelastic system with second sound

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Abstract

In this work we consider a strain gradient porous thermoelastic bar. We use the semigroup approach and Gearhart Theorem to prove the existence of a unique weak solution that decays exponentially.

Keywords: nonsimple material, second sound thermoelasticity. well-posedness, exponential decay.

1 Introduction

We consider the following problem,

$$\begin{cases} \rho u_{tt} = a u_{xx} + b\phi_x - c u_{xxxx} - d\phi_{xxx} - \delta\theta_x & in[0,\pi]\times[0,\infty[, \\ J\phi_{tt} = d u_{xxx} + \beta\phi_{xx} - \xi\phi - b u_x + m\theta - \mu\phi_t & in[0,\pi]\times[0,\infty[, \\ c^*\theta_t = -q_x - \delta u_{xt} - m\phi_t & in[0,\pi]\times[0,\infty[, \\ \tau q_t + q + \kappa\theta_x = 0. & in[0,\pi]\times[0,\infty[, \end{cases} \tag{1}$$

where u, ϕ, θ and q are respectively, the transversal displacement, the volume fraction, the difference of temperature from an equilibrium reference value and the heat flux of a one dimensional elastic material of length π. The coefficients $\rho, J, a, c, c^*, \beta, \xi, \tau, \mu$ and κ are positive constitutive constants.

We assume that the solutions verify the boundary and the initial conditions

$$\begin{aligned} &u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0 \quad t>0, \\ &\phi_x(t,0) = \phi_x(t,\pi) = q(t,0) = \theta(t,\pi) = 0 \qquad t>0, \end{aligned} \tag{2}$$

and

$$\begin{cases} u(0,x)=u_0(x),\ u_t(0,x)=v_0(x),\ \phi(0,x)=\phi_0(x), & \text{in } x\in[0,\pi],\\ \phi_t(0,x)=\varphi_0(x), \theta(0,x)=\theta_0(x), q(0,x)=q_0(x) & \text{in } x\in[0,\pi]. \end{cases} \tag{3}$$

The energy of system (1) is defined by

$$E(t):=\frac{1}{2}\int_0^\pi \Big[\rho|u_t|^2+a|u_x|^2+c|u_{xx}|^2+J\,|\phi_t|^2+\xi\,|\phi|^2+\beta\,|\phi_x|^2+c^*|\theta|^2+\frac{\tau}{\kappa}|q|^2+2bRe(u_x\phi)+2dRe(u_{xx}\phi_x)\Big]dx,$$

We assume that the matrix

$$A=\begin{pmatrix} a & b\\ b & \xi\end{pmatrix}, B=\begin{pmatrix} c & d\\ d & \beta\end{pmatrix}$$

are positive definites. Then, the energy $E(t)$ satisfies

$$\frac{dE}{dt}(t)=-\frac{1}{\kappa}\int_0^\pi |q|^2dx-\mu\int_0^\pi |\phi_t|^2\,dx.$$

2 Well-posedness

Introducing the new variables $v=u_t$ and $\psi=\phi_t$ the problem can be written

$$(u_t,v_t,\phi_t,\psi_t,\theta_t,q_t)^T=\mathcal{A}(u,v,\phi,\psi,\theta,q)^T$$

where $\mathcal{A}$ is the operator defined on

$$\mathcal{H}:=\Big\{(u,v,\phi,\psi,\theta,q)\in\Big(H^2(0,\pi)\cap H_0^1(0,\pi)\Big)\times L^2(0,\pi)\times H^1(0,\pi)\times L^2(0,\pi)\times L^2(0,\pi)\times L^2(0,\pi),$$
$$\int_0^\pi\phi(x)dx=\int_0^\pi\psi(x)dx=\int_0^\pi\theta(x)dx=\int_0^\pi q(x)dx=0\Big\},$$

with domain

$$D(\mathcal{A})=\Big\{U\in\mathcal{H}: u\in H^3(0,\pi), v\in H^2(0,\pi)\cap H_0^1(0,\pi), \phi\in H^2(0,\pi), \psi\in H^1(0,\pi), \theta\in H^1(0,\pi),$$
$$q\in H^1(0,\pi), (cu_x+d\phi)\in H^3(0,\pi), (du_x+\beta\phi)\in H^2(0,\pi)\Big\}.$$

Lemme 2.1. *The operator $\mathcal{A}$ defined in* (3) *is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

To establish the existence and the uniqueness of the solution of the system (1) we apply the theorem of Lumer-Phillips, we will show that $\mathcal{A}$ is a dissipative operator and $0\in\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$.

3 Exponential stability

Theorem 3.1. *A C_0-semigroup of contractions $S(t) = e^{-\mathcal{A}t}$, generated by an operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, is exponentially stable if and only if*

i) $i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A})$,

ii) $\overline{\lim}_{|\lambda|\to\infty} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$

To prove the exponential stability we use the following two Lemmas.

Lemme 3.1. $i\mathbb{R} = \{i\lambda; \lambda \in \mathbb{R}\}$ *is contained in* $\rho(\mathcal{A})$.

Proof. By use The operator $\mathcal{A}^{-1}$ is compact and the compact injection of $H^m(0;\pi)$ into $H^j(0;\pi)$ is for $m > j$, we get the subsequence $U_r = \mathcal{A}^{-1}F_r$ converges in $\mathcal{H}$

Suppose that there exists $\lambda \in \mathbb{R}(\lambda \neq 0)$ such that $i\lambda$ is in the spectrum of $\mathcal{A}$. Then, there exists a vector such that

$$(i\lambda I - \mathcal{A})U = 0$$

We prove that

$$U = 0$$

which contradiction and the proof is complete. □

Lemme 3.2. *The operator $\mathcal{A}$ defined in* (3) *satisfies*

$$\overline{\lim}_{|\lambda|\to\infty} \left\|(i\lambda I - \mathcal{A})^{-1}\right\| < \infty.$$

Proof. It suffices to prove that there exists a positive constant C such that

$$\|(i\lambda I - \mathcal{A})^{-1}U\| \leq C\|F\|, \lambda \in \mathbb{R}$$

□

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1-D Green function for an infinite well plus linear potential

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1 Abstract

In this work, we present a new result which concerns the derivation of the Green function relative to the time-independent Schrödinger equation in one-dimensional space. The system considered in this work is a quantal particle that moves in an infinite potential well plus a linear potential. At first, we have assumed that the particle moves in the interval [0,b]. Secondly, we have assumed that the particle moves in the interval [-b,b]. We have used the continuity of the solution and its derivative to obtain the associate Green function showing the discrete spectra of the Hamiltonian. A comparison of eigen values q between the two cases has been reported.

Problem (1):
$$\begin{cases} y'' + (q - V(x))\, y = 0 \\ V(x) = \begin{cases} \infty, &]-\infty, 0[\, U\,]b, +\infty[\\ \lambda x, & 0 \leq x \leq b \end{cases} \\ \text{avec } y(0) = 0 \text{ et } y(b) = 0 \\ \lambda \geq 0 \end{cases}$$

Problem (2):
$$\begin{cases} y'' + (q - V(x))\, y = 0 \\ V(x) = \begin{cases} \infty, &]-\infty, -b[\, U\,]b, +\infty[\\ \lambda x, & -b \leq x \leq b \end{cases} \\ \text{avec } y(-b) = 0 \text{ et } y(b) = 0 \\ \lambda \geq 0 \end{cases}$$

Global existence, asymptotic stability and numerical simulation for a coupled two–cell Schnakenberg reaction–diffusion model

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Abstract

This paper deals with a coupled two-cell Schnakenberg reaction-diffusion system subject to Neumann boundary conditions. Firstly, we obtain the global existence and uniqueness of classical solution for the system and under certain conditions for the parameters, we obtain the asymptotic stability. Then, with the aim of displaying the dynamics of suggested model, we develop a positivity preserving splitting technique to find the numerical solution of the proposed system.

Keywords: Reaction–diffusion system; existence of solution; asymptotic stability; linearization; Numerical method.

1. Introduction

The Schnakenberg model is a simple system of reaction-diffusion equations occurring in chemical kinetics and biological processes. It was first proposed by Schnakenberg in 1979 (cf. [1]). The Schnakenberg chemical reaction can be represented by the following mechanism:

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} U, \quad B \overset{k_2}{\to} V, \quad 2U + V \overset{k_3}{\to} 3U, \tag{1.1}$$

where A, B, U and V are chemical reactants and products. The steps in (1.1) yield the nondimensionalized Schnakenberg reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + a - u + u^2 v, & x \in (0,l), t > 0 \\ \frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + b - u^2 v, & x \in (0,l), t > 0 \end{cases} \tag{1.2}$$

subject to the homogeneous Dirichlet or Neumann boundary conditions and initial data. In this system, the reactions occur in an interval $(0,l)$, $l > 0$, $u := u(x,t)$ and $v := v(x,t)$ are the chemical concentrations of an activator and an inhibitor, respectively, and d_1, d_2, a, b are positive numbers. The Schnakenberg system (1.2) has drawn the attention of various researchers who have obtained interesting analytical and numerical results [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In general, numerous numerical schemes have been proposed for solving evolutionary partial differential equations [13, 14, 15, 16, 17, 18, 19].

In this work, we consider a coupled two-cell Schnakenberg model

$$\begin{cases} u_t - d_1 \Delta u = a - u + u^2 v + c(w - u) & \text{in } \Omega \times (0,T) \\ v_t - d_2 \Delta v = b - u^2 v & \text{in } \Omega \times (0,T) \\ w_t - d_3 \Delta w = a - w + w^2 z + c(u - w) & \text{in } \Omega \times (0,T) \\ z_t - d_4 \Delta z = b - w^2 z & \text{in } \Omega \times (0,T) \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_\nu z = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{on } \bar{\Omega} \\ w(x,0) = w_0(x), z(x,0) = z_0(x) & \text{on } \bar{\Omega} \end{cases} \tag{1.3}$$

where $u := u(x,t)$, $v := v(x,t)$, $w := w(x,t)$, $z := z(x,t)$, $\Omega := (L_{in}, L_{end})$, $L_{in} \geqslant 0$, d_1, d_2, d_3, d_4, a, b, c, L_{end} and T are positive numbers, Δ is the Laplacian operator on Ω, and ν is the unit outer normal to $\partial\Omega$. Some results have been reported regarding the dynamics of the coupled two-cell Schnakenberg model (1.3) with/without modifications on the system [10, 12].

In this paper, we are primarily interested in continuing the work on the two-cell Schnakenberg model (activator-inhibitor system), in which we demonstrate the global existence of classical solution for system (1.3) by using a suitable Lyapunov-type function (cf. J. Morgan [20]). In addition, we study the local asymptotic stability and develop an efficient explicit unconditionally numerical scheme which preserves positivity of the solution for system (1.3). To affirm our findings, numerical examples will be presented.

2. Global existence of solutions

The principal step toward the result is to establish a so-called Lyapunov-type function $H \in C^2(\mathbb{R}_+^4; \mathbb{R})$ with $h_i \in C^2(\mathbb{R}_+; \mathbb{R})$ for $i = 1, \ldots, 4$ such that

$$H(z) = \sum_{i=1}^{4} h_i(z_i), \quad z = (z_i)_{i=1}^4 \in \mathbb{R}_+^4, \tag{2.1}$$

with

$$h_i(z_i), h_i''(z_i) \geq 0, \quad z_i \in \mathbb{R}_+, \quad i = 1, \ldots, 4, \tag{2.2}$$

and

$$H(z) \longrightarrow \infty, \quad \text{if and only if} \quad |z| \longrightarrow \infty \text{ in } \mathbb{R}_+^4, \tag{2.3}$$

Assume there exists a lower triangular matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant 4} \in \mathbb{R}^4 \times \mathbb{R}^4$ which satisfies $a_{ij} \geqslant 0$, $a_{ii} > 0$ for $1 \leqslant i, j, \leqslant 4$ such that for all $1 \leqslant l \leqslant 4$ there exist $K_1, K_2 \geqslant 0$ independent of l, in which

$$\sum_{j=1}^{l} a_{jl} h_j'(z_j) f_j(z) \leq K_1 H(z) + K_2, \quad z \in \mathbb{R}_+^4. \tag{2.4}$$

Also, assume that there exist $q, K_3, K_4 \geqslant 0$ such that for $1 \leqslant i \leqslant 4$,

$$h_i'(z_i) f_i(z) \leq K_3 (H(z))^q + K_4, \quad z \in \mathbb{R}_+^4. \tag{2.5}$$

Furthermore, assume that there exist $K_5, K_6 \geqslant 0$ such that

$$\nabla H(z).f(z) \leq K_5 H(z) + K_6, \quad z \in \mathbb{R}_+^4. \tag{2.6}$$

Based on J. Morgan's method (cf. [20]), we get the following result

Theorem 1. *Consider the initial condition $(u_0, v_0, w_0, z_0) \in [L^\infty(\Omega; (0, +\infty))]^4$. Then, there exists a unique positive global (i.e. $T = \infty$) classical solution for the coupled two-cell Schnakenberg system (1.3).*

Proof 1. *At the outset, we swap variables u and v as well as w and z in the coupled two-cell Schnakenberg system (1.3), which yields the equivalent system*

$$\begin{cases} u_t - d_1 \Delta u = f_1(U) & \text{in } \Omega \times (0,T) \\ v_t - d_2 \Delta v = f_2(U) & \text{in } \Omega \times (0,T) \\ w_t - d_3 \Delta w = f_3(U) & \text{in } \Omega \times (0,T) \\ z_t - d_4 \Delta z = f_4(U) & \text{in } \Omega \times (0,T) \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_\nu z = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{on } \bar{\Omega} \\ w(x,0) = w_0(x), z(x,0) = z_0(x) & \text{on } \bar{\Omega} \end{cases} \tag{2.7}$$

where $U := (u, v, w, z)$ *and*

$$\begin{aligned} f_1(U) :=& b - uv^2, \\ f_2(U) :=& a - v + uv^2 + c(z - v), \\ f_3(U) :=& b - wz^2, \\ f_4(U) :=& a - z + wz^2 + c(v - z). \end{aligned}$$

The local existence of a solution for system (1.3) results from the well-known semigroup theory (cf. [23, 24]). We choose $h_i(z_i) = z_i$ *for* $i = 1, \ldots 4$, *and* $A = (a_{ij})_{1 \leqslant i,j \leqslant 4}$ *such that*

$$a_{ij} := \begin{cases} 1 & if \quad i \geqslant j \\ 0 & else \end{cases}$$

Then, conditions (2.1)-(2.6) are fulfilled.

Remark 1. *The proof of Theorem 1 remains effective when* Ω *in an open subset of* $\mathbb{R}^n (n \in \mathbb{N})$ *that is both smooth and bounded (cf. [20]). The same results can also be obtained with a wider class of boundary conditions by using the method of S. Abdelmalek and S. Kouachi (cf. [25]).*

3. Stability analysis

In this section, we study the local stability of the unique constant solution

$$(u^*, v^*, w^*, z^*) = \left(a + b, \frac{b}{(a+b)^2}, a + b, \frac{b}{(a+b)^2} \right)$$

of system (1.3).

Theorem 2. *Subject to the conditions* $d_1 = d_3$, $d_2 = d_4$ *and* $a > b$, *the positive constant solution* (u^*, v^*, w^*, z^*) *of system* (1.3) *is uniformly asymptotically stable.*

Proof 2. *Let* $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ *be the eigenvalues of the operator* $-\Delta$ *on* Ω *with the homogeneous Neumann boundary condition. Set* $\mathbf{X}_j$ *is the eigenspace corresponding to* μ_j. *Let*

$$\mathbf{X} = \left\{ (u, v, w, z) \in \left[C^1(\bar{\Omega}) \right]^4 \mid \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_v z = 0 \text{ on } \partial\Omega \right\}$$

$\{\phi_{jl}; l = 1, \ldots, m(\mu_j)\}$ *be an orthonormal basis of* $\mathbf{X}_j$, *and* $\mathbf{X}_{jl} = \left\{ \mathbf{c}\phi_{jl} \mid \mathbf{c} \in \mathbf{R}^4 \right\}$. *Here* $m(\mu_j)$ *is the multiplicity of* μ_j. *Then*

$$\mathbf{X} = \bigoplus_{j=0}^{\infty} \mathbf{X}_j \text{ and } \mathbf{X}_j = \bigoplus_{l=1}^{m(\mu_j)} \mathbf{X}_{jl}.$$

The linearization of (1.3) *at* (u^*, v^*, w^*, z^*) *is*

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = L \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} + \begin{pmatrix} g_1(u - u*, v - v*, w - w*, z - z*) \\ g_2(u - u*, v - v*, w - w*, z - z*) \\ g_3(u - u*, v - v*, w - w*, z - z*) \\ g_4(u - u*, v - v*, w - w*, z - z*) \end{pmatrix},$$

where $g_i(y_1, y_2, y_3, y_4) = O(y_1^2, y_2^2, y_3^2, y_4^2)$*, for* $i = 1, 2, 3, 4$*, and*

$$L = \begin{pmatrix} d_1\Delta + \frac{2b}{a+b} - 1 - c & (a+b)^2 & c & 0 \\ -\frac{2b}{a+b} & d_2\Delta - (a+b)^2 & 0 & 0 \\ c & 0 & d_1\Delta + \frac{2b}{a+b} - 1 - c & (a+b)^2 \\ 0 & 0 & -\frac{2b}{a+b} & d_2\Delta - (a+b)^2 \end{pmatrix}.$$

For each j $(j = 0, 1, 2, ...)$, $\mathbf{X}_j$ *is invariant under the operator* L*, and* ξ_j *is an eigenvalue of the operator* L *on* $\mathbf{X}_j$ *if and only if* ξ_j *is aneigenvalue of the following matrix*

$$A_j = \begin{pmatrix} -d_1\mu_j + \frac{2b}{a+b} - 1 - c & (a+b)^2 & c & 0 \\ -\frac{2b}{a+b} & -d_2\mu_j - (a+b)^2 & 0 & 0 \\ c & 0 & -d_1\mu_j + \frac{2b}{a+b} - 1 - c & (a+b)^2 \\ 0 & 0 & -\frac{2b}{a+b} & -d_2\mu_j - (a+b)^2 \end{pmatrix}.$$

Denote

$$B_j^{(1)} = \begin{pmatrix} -d_1\mu_j + \frac{2b}{a+b} - 1 & -2b \\ a+b & -d_2\mu_j - (a+b)^2 \end{pmatrix},$$

$$B_j^{(2)} = \begin{pmatrix} -d_1\mu_j + \frac{2b}{a+b} - 1 - 2c & -2b \\ a+b & -d_2\mu_j - (a+b)^2 \end{pmatrix}.$$

It is easy to see that ξ_j *is an eigenvalue of* A_j *if and only if* ξ_j *is an eigenvalue of* $B_j^{(1)}$ *or* $B_j^{(2)}$*.*

In order to analyze the eigenvalue of L*, it is sufficient to analyze the eigenvalue of* $B_j^{(1)}$ *and* $B_j^{(2)}$*. We first consider the matrix* $B_j^{(1)}$*. Simple computations yield*

$$\begin{aligned} det\left(B_j^{(1)}\right) &= \left(-d_1\mu_j + \frac{2b}{a+b} - 1\right)\left(-d_2\mu_j - (a+b)^2\right) + 2b\,(a+b) \\ &= \mu_j\left(d_1 d_2\mu_j + d_2\left(1 - \frac{2b}{a+b}\right) + d_1\,(a+b)^2\right) + a - b \\ &= \mu_j\left(d_1 d_2\mu_j + d_2\frac{a-b}{a+b} + d_1\,(a+b)^2\right) + a - b \\ Tr\left(B_j^{(1)}\right) &= -\,(d_1 + d_2)\,\mu_j - \left(\frac{-2b}{a+b} + 1\right) - (a+b)^2 \\ &= -\,(d_1 + d_2)\,\mu_j - \frac{a-b}{a+b} - (a+b)^2 \end{aligned}$$

where $det\left(B_j^{(1)}\right)$ *and* $Tr\left(B_j^{(1)}\right)$ *are respectively the determinant and trace of* $B_j^{(1)}$*.*

As above, we then consider the $B_j^{(2)}$. In view of simple computations, we get the following

$$\begin{aligned}
det\left(B_j^{(2)}\right) &= \left(-d_1\mu_j + \frac{2b}{a+b} - 1 - 2c\right)\left(-d_2\mu_j - (a+b)^2\right) + 2b\,(a+b) \\
&= \mu_j\left(d_1 d_2 \mu_j + d_2\left(1 + 2c - \frac{2b}{a+b}\right) + d_1\,(a+b)^2\right) + a - b + 2c\,(a+b)^2 \\
&= \mu_j\left(d_1 d_2 \mu_j + d_2\left(\frac{a-b}{a+b} + 2c\right) + d_1\,(a+b)^2\right) + a - b + 2c\,(a+b)^2 \\
Tr\left(B_j^{(2)}\right) &= -\,(d_1 + d_2)\,\mu_j - \left(\frac{-2b}{a+b} + 1 + 2c\right) - (a+b)^2 \\
&= -\,(d_1 + d_2)\,\mu_j - \left(\frac{a-b}{a+b} + 2c\right) - (a+b)^2
\end{aligned}$$

Thus, subject to the condition $a > b$ we get that, $det\left(B_j^{(i)}\right) > 0$ and $Tr\left(B_j^{(i)}\right) < 0$, for $j = 0, 1, 2, 3, \ldots$ and $i = 1, 2$. Hence, the positive constant solution (u^*, v^*, w^*, z^*) of system (1.3) is uniformly asymptotically stable (cf. [21]).

4. Numerical method

4.1. Discretization of domain

We aim to show the approximate solutions of system (1.3) by applying the finite difference method. We consider the numerical approximation in the time domain $[0, T]$ and the space domain $[L_{in}, L_{end}]$. Let $t_k = k\Delta t$ $(0 \leqslant t_k \leqslant T)$, $k = 0, \ldots, M$, $x_i = L_{in} + i\Delta x$ $(L_{in} \leqslant x_i \leqslant L_{end})$, and $i = 0, \ldots, N$, where the time step is $\Delta t = \frac{T}{M}$ and the space step is $\Delta x = \frac{L_{end} - L_{in}}{N}$ $(M, N \in \mathbb{N})$. We use the following notations $u_i^k = u(x_i, t_k)$, $v_i^k = v(x_i, t_k)$, $w_i^k = w(x_i, t_k)$, $z_i^k = z(x_i, t_k)$.

4.2. Numerical scheme

Based on Mickens' rules for the nonstandard finite difference schemes (cf. [26]), we obtain the following new scheme for system (1.3):

$$\begin{aligned}
u_i^{k+1} = & u_i^k + \alpha_1\left(u_{i-1}^k + u_{i+1}^k\right) + a\Delta t - \Delta t u_i^{k+1} + \Delta t (u_i^k)^2 v_i^k \\
& + c\Delta t w_i^k - c\Delta t u_i^{k+1} - 2\alpha_1 u_i^{k+1}, \qquad (4.1) \\
v_i^{k+1} = & v_i^k + \alpha_2\left(v_{i-1}^k + v_{i+1}^k\right) - 2\alpha_2 v_i^{k+1} + b\Delta t - \Delta t (u_i^k)^2 v_i^{k+1}, \qquad (4.2) \\
w_i^{k+1} = & w_i^k + \alpha_3\left(w_{i-1}^k + w_{i+1}^k\right) + a\Delta t - \Delta t w_i^{k+1} + \Delta t (w_i^k)^2 z_i^k \\
& + c\Delta t u_i^k - c\Delta t w_i^{k+1} - 2\alpha_3 w_i^{k+1}, \qquad (4.3) \\
z_i^{k+1} = & z_i^k + \alpha_4\left(z_{i-1}^k + z_{i+1}^k\right) - 2\alpha_4 z_i^{k+1} + b\Delta t - \Delta t (w_i^k)^2 z_i^{k+1}, \qquad (4.4) \\
& \qquad (4.5)
\end{aligned}$$

where $\alpha_j = \frac{d_j \Delta t}{(\Delta x)^2}$ for $j = 1, \ldots, 4$. Thus, the explicit formula is

$$u_i^{k+1} = \frac{u_i^k + \alpha_1 \left(u_{i-1}^k + u_{i+1}^k\right) + a\Delta t + \Delta t (u_i^k)^2 v_i^k + c\Delta t w_i^k}{1 + 2\alpha_1 + \Delta t(c+1)}, \tag{4.6}$$

$$v_i^{k+1} = \frac{v_i^k + \alpha_2 \left(v_{i-1}^k + v_{i+1}^k\right) + b\Delta t}{1 + 2\alpha_2 + \Delta t (u_i^k)^2}, \tag{4.7}$$

$$w_i^{k+1} = \frac{w_i^k + \alpha_3 \left(w_{i-1}^k + w_{i+1}^k\right) + a\Delta t + \Delta t (w_i^k)^2 z_i^k + c\Delta t u_i^k}{1 + 2\alpha_3 + \Delta t(c+1)}, \tag{4.8}$$

$$z_i^{k+1} = \frac{z_i^k + \alpha_4 \left(z_{i-1}^k + z_{i+1}^k\right) + b\Delta t}{1 + 2\alpha_4 + \Delta t (w_i^k)^2}. \tag{4.9}$$

4.3. Positivity, stability and consistency of the proposed scheme

By following the arguments in [22] it is not difficult to obtain the following results:

Theorem 3. *Subject to the initial conditions of system (1.3) being non-negative, the numerical scheme (4.6)-(4.9) demonstrates positive solutions.*

Theorem 4. *Subject to the initial conditions of system (1.3) being non-negative, the numerical scheme (4.6)-(4.9) is stable.*

Theorem 5. *Subject to the initial conditions of system (1.3) being non-negative, the numerical scheme (4.6)-(4.9) is consistent.*

5. Numerical experiments

Let us now show the approximate solution of the coupled two-cell Schnakenberg system (1.3) in order to demonstrate the changes in solution behaviour that arise when the parameters are varied. The computer algorithm for numerical scheme (4.6)-(4.9) was written in Matlab. Throughout the simulations we considered the following initial conditions:

$$\begin{aligned} u_0(x) &= a + b + 1 + \frac{\cos(1.3\pi x)}{8}, \\ v_0(x) &= a + 1 + \frac{\sin(10\pi x)}{5}, \\ w_0(x) &= \frac{a + b + 1.7}{2} + \frac{\cos(5\pi x)}{8}, \\ z_0(x) &= \frac{b + 0.2}{2} + \frac{\sin(5\pi x)}{10}. \end{aligned}$$

The following sets of system parameters are considered for each simulation:

1. $a = 1.7$, $b = 1.5$, $c = 2$, $L_{in} = 0$, $L_{end} = 2$, $T = 200$, $d_1 = 0.01$, $d_2 = 0.02$, $d_3 = 0.0.01$ and $d_4 = 0.0.02$.

2. $a = 0.042$, $b = 0.626$, $c = 2$, $L_{in} = 0$, $L_{end} = 2$, $T = 200$, $d_1 = 0.01$, $d_2 = 0.02$, $d_3 = 0.0.01$ and $d_4 = 0.0.02$.

Remark 2. *The approximate solution depicted in Figure 1 agree with the theoretical results obtained previously regarding the dynamics of system* (1.3)*.*

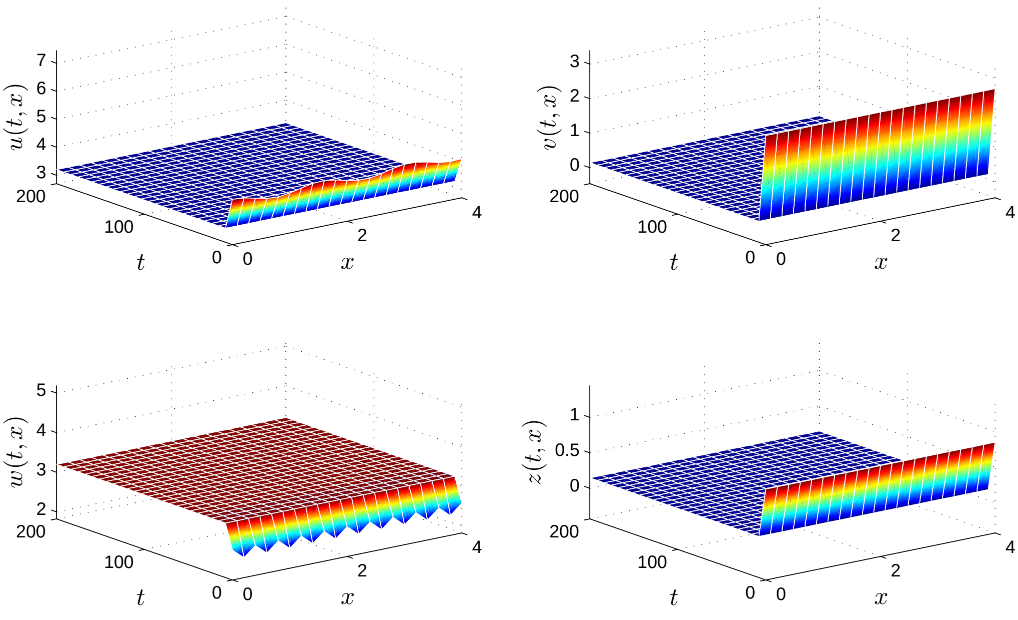

Figure 1: Numerical simulation of a coupled two-cell Schnakenberg system (1.3) subject to the first set of parameters.

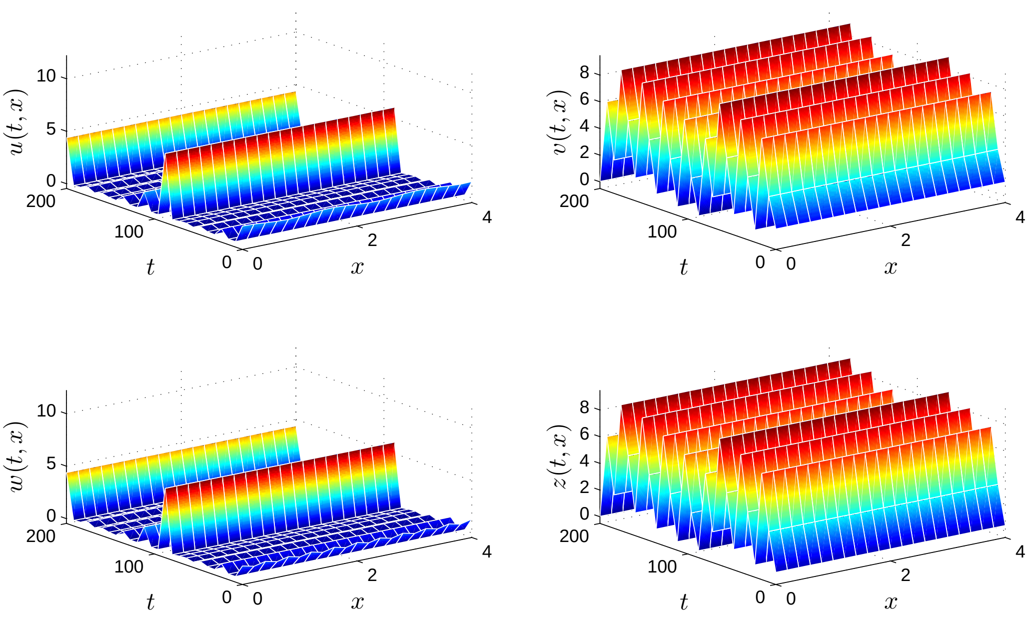

Figure 2: Numerical simulation of a coupled two-cell Schnakenberg system (1.3) subject to the second set of parameters.

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ETUDE VARIATIONNELLE D'UN PROBLÈME DE CONTACT AVEC USURE ET ENDOMMAGEMENT

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ABSTRACT. L'objet de ce travail est l'étude théorique d'un système non linéaire aux dérivées patielles modélisant un problème mécanique. Le système considéré modilise un problème de contact avec frottement et usure entre un corps électro-mécanique et fondation. Les résultats obtenus concernent l'existence et l'unicité de la solution faible. Le s méthodes employées sont basées sur la théorie des équations, inéquations variationelles et quasivariationelles.

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ANALYSIS OF QUASISTATIC VISCOELASTIC VISCOPLASTIC PIEZOELECTRIC CONTACT PROBLEM WITH FRICTION AND ADHESION

NADHIR CHOUGUI

ABSTRACT. In this paper we study the process of bilateral contact with adhesion and friction between a piezoelectric body and an insulator obstacle, the so-called foundation. The material's behavior is assumed to be electro-viscoelastic-viscoplastic; the process is quasistatic, the contact is modeled by a general nonlocal friction law with adhesion. The adhesion process is modeled by a bonding field on the contact surface. We derive a variational formulation for the problem and then, under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution to the model.The proofs are based on a general results on elliptic variational inequalities and fixed point arguments.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 74M10, 74M15, 74F05, 74R05, 74C10.

KEYWORDS AND PHRASES. Viscoelastic, viscoplastic, piezoelectric, bilateral contact, non local Coulomb friction, Adhesion, Quasi-variational inequality, Weak solution, fixed point.

1. DEFINE THE PROBLEM

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal ν. The body is acted upon by body forces of density f_0 and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\ \Gamma_1 > 0$ and $meas\ \Gamma_a > 0$. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Surface tractions of density f_2 act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in adhesive and frictional contact with an insulator obstacle, the so-called foundation.

We are interested in the deformation of the body on the time interval $[0, T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on x or t. Here and everywhere in this paper, i, j, k, $l = 1, ..., d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of x. The dot above variable represents the time derivatives.

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ and by "$.$", $\|.\|$ the inner product and the norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $u.\upsilon = u_i\upsilon_i$, $\|\upsilon\| = (\upsilon.\upsilon)^{1/2}$ for $u = (u_i)$, $\upsilon = (\upsilon_i) \in \mathbb{R}^d$, and $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\sigma\| = (\sigma.\sigma)^{1/2}$ for $\sigma = (\sigma_{ij})$, $\tau = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $\upsilon_\nu = \upsilon \cdot \nu$, $\upsilon_\tau = \upsilon - \upsilon_\nu\nu$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$, and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$. With these assumptions, the classical model for the process is the following.

Problem ($\mathcal{P}$). *Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Omega \times [0, T] \to \mathbb{R}$ such that*

$$\sigma(x,t) = \mathcal{A}\varepsilon(\dot{u}(x,t)) + \mathcal{F}\varepsilon(u(x,t)) + \int_0^t \mathcal{G}(\sigma(x,s), \varepsilon(u(x,s))ds - \mathcal{E}^*\mathbf{E}(\varphi(x,t)) \qquad in\ \Omega \times (0,T), \tag{1}$$

$$D = \mathcal{B}\mathbf{E}(\varphi) + \mathcal{E}\varepsilon(u) \qquad in\ \Omega \times (0,T), \tag{2}$$

$$Div\sigma + f_0 = 0 \qquad in\ \Omega \times (0,T), \tag{3}$$

$$divD = q_0 \qquad in\ \Omega \times (0,T), \tag{4}$$

$$u = 0 \qquad on\ \Gamma_1 \times (0,T), \tag{5}$$

$$\sigma\nu = f_2 \qquad on\ \Gamma_2 \times (0,T), \tag{6}$$

$$u_\nu = 0, \qquad on\ \Gamma_3 \times (0,T), \tag{7}$$

$$\begin{cases} \bullet\ \|\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)\| \leq \mu p(|R\sigma_\nu|), \\ \bullet\ \|\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)\| < \mu p(|R\sigma_\nu|) \\ \quad \Rightarrow \dot{u}_\tau = 0, \\ \bullet\ \|\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)\| = \mu p(|R\sigma_\nu|) \\ \quad \Rightarrow \exists\ \lambda > 0, such that : \\ \quad \sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau) = -\lambda\ \dot{u}_\tau, \end{cases} \qquad on\ \Gamma_3 \times (0,T), \tag{8}$$

$$\dot{\beta}(t) = -(\beta(t)\gamma_\tau\|R_\tau(u_\tau(t))\|^2 - \varepsilon_a)_+ \qquad on\ \Gamma_3 \times (0,T), \tag{9}$$

(10)

$$\varphi = 0 \qquad on\ \Gamma_a \times (0,T), \tag{11}$$

$$D.\nu = q_2 \qquad on\ \Gamma_b \times (0,T), \tag{12}$$

$$D.\nu = 0 \qquad on\ \Gamma_3 \times (0,T), \tag{13}$$

$$u(0) = u_0 \qquad in\ \Omega, \tag{14}$$

$$\beta(0) = \beta_0 \qquad on\ \Gamma_3. \tag{15}$$

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Uniform Convergence of Multigrid Methods for Variational Inequalities

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Abstract

In this paper, we will apply the Multi-Grid Method for Variational Inequalities, in the measure where the obstacle depond of the solution. Moreover, we prove the uniforme convergence of this multi-grid algorithm with Gauss-Seidel's iteration as smoothing procedure.

Keywords :

Variational inequality, Quasi-variational elliptic inequality, Multigrids methods, finite differences, finite element, Approximations.

1 Introduction

For our workIn this paper we will stady the multi-grid method for the quasi-variational inequalities, in the measure where the obstacle depond of the solution.
We are goin to present our work according to the following outline: in the second section, we will give our continuous problem. after that we will apply on finite element discretizations and introduce an algorithm of this problem.which gives a formulation in stationary Hamilton Jacobi Bellman equations inspired by Hoppe multigrid method [2]. Finally, we give results for the approximation and smoothing properties in L_∞ norm and prove the uniform convergence of the multi-grid algorithm.

1.1 The continuous problem

Let Ω be an open in R^n, with sufficiently smooth boundary $\partial\Omega$ for $u, v \in H^1(\Omega)$, consider the bilinear form as follows:

$$a(u,v)=\int\limits_{\Omega}[\sum_{1\leq i,j\leq n}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{1\leq i,j\leq n}a_i(x)\frac{\partial u}{\partial x_i}+a_0(x)u\cdot v]dx \tag{1}$$

- Where $a_{ij}\left(x\right), a_i\left(x\right), a_0(x), x\in\overline{\Omega}, 1\leq i,j\leq n$ are sufficiently smooth coefficients and satisfy the following conditions:

$$\sum_{1\leq i,j\leq n}a_{ij}\psi_i\psi_j\geq \upsilon|\psi|^2, \psi\in R^n, \upsilon>0$$

$$a_0\left(x\right)\geq\beta>0,$$

-Where β is a constant.and M is operator given by:

$$Mu=k+\inf_{\varepsilon\geq 0, x+\varepsilon\in\overline{\Omega}}u\left(x+\varepsilon\right) \tag{2}$$

-Where $k>0$ and M satisfies

$$Mu\in W^{2,p}\left(\Omega\right), Mu\geq 0, on\ \partial\Omega: 0\leq g\leq Mu$$

-Where g is a regular function defined on $\partial\Omega$.
-$K_g(u)$, remove an implicit convex and non empty set which defined as follows

$$K_g=\left\{v\in H^1\left(\Omega\right), v=g\ \ on\ \ \partial\Omega, v\leq Mu, in\ \Omega\right\} \tag{3}$$

We consider the following problem: Find $u\in K_g$ the solution of

$$\begin{cases} a\left(u, v-u\right)\geq\left(f, v-u\right) \qquad in\ \Omega, v\in K_g(u) \\ u\leq Mu; \qquad Mu\geq 0 \\ u=g; \qquad on\ \ \partial\Omega. \end{cases} \tag{4}$$

1.2 The discrete problem

We denote by V_h the standard piecewise linear finite element space (where V_h form an internal approximation), we consider the discrete quasi-variational inequality Find $u_h\in K_{gh}$ such that

$$\begin{cases} a\left(u_h, v-u_h\right)\geq\left(f, v-u_h\right) \qquad \forall u_h, v\in K_{gh} \\ u_h\leq R_hMu_h; \qquad Mu\geq 0 \\ u_h=\pi_hg; \qquad on\ \ \partial\Omega. \end{cases} \tag{5}$$

Were $f\in L^\infty\left(\Omega\right), Mu=k+\inf\limits_{\varepsilon\geq 0, x+\varepsilon\in\overline{\Omega}}u_h\left(x+\varepsilon\right)$
And $K_{gh}=\{v\in V_h, v=\pi_hg\ on\ \partial\Omega, v\leq R_hMu_h, in\ \Omega\}$

π_h denote the interpolation operator on $\partial\Omega$ and R_h is the usual finite element restriction operator on Ω.

2 Description of the Multigrid Method for VIs

Let h_k be the discretization step over Ω. The finite element discretization conventionally leads to the discrete IV solution in finite dimension:

Find $u_k \in V_k$ such as:

$$\begin{cases} \langle A_k u_k, v_k - u_k \rangle \geq \langle f, v_k - u_k \rangle, & \forall v_k \in V_k \\ u_k \leq r_k M u_k, & v_k \leq r_k M u_k \end{cases} \tag{6}$$

put an iterated $u_k^v, v > 0$, we first determine $\bar{u}_k^v$ by p_k applications of a relaxation method

Note that:

$$\bar{u}_k^v = S_k^{p_k}(u_k^v) \tag{7}$$

or :

S_k is the iteration or smoothing operator

p_k is the number of iterations performed.

it is clear to verify that the IVs (7) are equivalent to the following PCNs.

Find $u_k \in \mathbb{R}^{n_k}$ solution of

$$\begin{cases} A_k u_k^* \leq F_k, \ u_k^* \leq r_k M u_k \\ \langle A_k u_k^* - F_k, u_k^* - r_k M u_k \rangle = 0 \end{cases} \tag{8}$$

Let us pose:

$$d_h^{(\nu)} = A_k \bar{u}_k^j - F_k, \quad \textit{le résidu de } u_k^v \tag{9}$$

It is immediate that the solution u_k^* of the problem (9) at the level k satisfies the following complementary problem:

$$\begin{cases} A_k u_k^* \leq A_k^v u_k - d_k^{(v)}, \ u_k^* \leq r_k M u_k \\ \left\langle A_k u_k^* - A_k \bar{u}_k^v + d_k^{(v)}, u_k^* - r_k M u_k \right\rangle = 0 \end{cases} \tag{10}$$

So to determine u_k completely, we need the calculate u_{k-1} at level $(k-1)$ as being the solution of:

$$\begin{cases} A_{k-1} u_{k-1} \leq g_{k-1}, \ u_{k-1} \leq r_k M u_k \\ \langle A_{k-} u_{k-1} - g_{k-1}, u_{k-1} - r_k M u_k \rangle = 0 \end{cases} \tag{11}$$

Where :

$$g_{k-1} = A_k r.\bar{u}_k^v - r.d_k^{(v)} \tag{12}$$

and r is the natural restriction

$$r = r_{k-1}^{-1}.r_k \tag{13}$$

we can interpret $u_{k-1} - r_k^{k-1}\bar{u}_k^v$ as an approximation at the level $k-1$ of the error $u_k^* - \bar{u}_k^\nu$.

Consequently, using an appropriate prolongation $p_{k-1}^{k} : R^{N_{k-1}} \to R^{N_k}$ we determine an improved iteration at the level k by

$$u_k^{v+1} = \bar{u}_k^{\nu} + p_{k-1}^{k}\left(u_{k-1} - r_k^{k-1}\bar{u}_k^{v}\right) \tag{14}$$

We are going to state a theorem of existence and unicity for the solution of the problems (4) and (5), and we will prove the convergence of Multigrid method for our problem (5).

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Estimating the conditional tail expectation for randomly censored heavy-tailed data

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Abstract The conditional tail expectation (CTE) has the advantage, over the very popular Value-at-Risk, of being a coherent risk measure. Hence, it has become a very useful tool in financial and actuarial risk assessment. For such quantity, [1] discussed the sample estimator and [2] proposed an estimator for an important class of Pareto-like distributions. In this paper, we consider data that are heavy-tailed and, at the same time, randomly censored. By making use of survival and extreme value methodologies, we define an estimator for the CTE and we construct confidence intervals and discuss their lengths and coverage probabilities. Finally, we apply our results to a set of real data, namely the survival times of Australian male Aids.

Keywords: Coherent risk measure; Conditional tail expectation; Extreme value index; Heavy-tails; Hill estimator; Kaplan-Meier estimator.

1 Simulation study

We carry out a simulation study to illustrate the performance of our estimator, through two sets of data from Burr (ξ, η) and Fréchet (ξ) models respectively defined, for $x \geq 0$, by

$$\bar{F}(x) = \left(1 + x^{1/\eta}\right)^{-\eta/\xi} \text{ and } \bar{F}(x) = 1 - \exp(-x^{-1/\xi}),$$

where ξ and η are two positive parameters.

The confidence intervals are constructed by the technique bootstrap, we use the percentile confidence intervals method.

2 Case study

In this section, we apply our estimation procedure to the dataset known as Australian Aids data and provided by Dr P.J. Solomon and the Australian National Centre in HIV Epidemiology and Clinical Research. It consists in medical observations on 2843 patients (among whom 2754 are male) diagnosed with Aids in Australia before July 1^{st}, 1991 (See [3] and [4]).

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Abstract

In this recherche, we study the existence and uniqueness of the solution of one problem of mechanical contact, exactly the elasto-viscoplastic contact with an internal state variable wich is modeled by a normal compliance condition without friction. Uses the fixed point theoremand order-one evolution equations with monotonic operators.

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NON-EXISTENCE OF GLOBAL SOLUTIONS OF NON-LINEAR FRACTIONAL REACTION-DIFFUSION SYSTEM OF INEQUALITIES ON THE HEISENBERG GROUP

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ABSTRACT. This paper deals with the non-existence result for solutions to the problem:

$$(FRDS):\begin{cases}\mathbf{D}^{\alpha}_{0/t}u-\left(\Delta_{\mathbb{H}}^{\frac{\beta}{2}}\right)(u) & \geq & \mid v\mid^{p}\\ \mathbf{D}^{\delta}_{0/t}v-\left(\Delta_{\mathbb{H}}^{\frac{\beta}{2}}\right)(v) & \geq & \mid u\mid^{q}\end{cases}$$

where $\mathbf{D}^{\alpha}_{0/t}$ is the time-fractional derivative of order $\alpha,\delta\in(0,1)$ in the sense of Caputo, $(\Delta_{\mathbb{H}})^{\beta/2}$ is the fractional Laplacian of order $\beta/2$ with $1<\beta\leq 2$ in the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}^N$. These non existence result Related to $Q\equiv 2N+2$ less than the critical exponents that depends on α, δ, β ,$p,q>1$, and $m\in\mathbb{N}$. For $\alpha=1$ we generalize and improve the results obtained in [4] from the parabolic system.

1. Introduction

Pohozaev and Véron [25] have established the question of nonexistence results for solutions of semilinear parabolic inequalities of the type:

$$(FPI):\frac{\partial u}{\partial t}-\Delta_{\mathbb{H}}(au)\geq|u|^{p}\tag{1.1}$$

it is shown that no weak solution u exists provided that:

$$\int_{\mathbb{R}^{2N+1}}u_0(\eta)d\eta\geq 0\quad\text{and}\quad 1<p\leq\frac{Q+2}{Q}\tag{1.2}$$

Their result have been generalized by B.Ahmed et all [1] to equation of the form:

$$(NLPE):\frac{\partial u}{\partial t}-(\Delta_{\mathbb{H}})^{\beta/2}\left(|u|^{m}\right)=|u|^{p}\tag{1.3}$$

where they proved that the equation (NLPE) admits no solution defined in $\mathbb{H}^N$ whenever

$$1<p<m+\frac{\beta}{Q}\quad\text{and}\quad\int_{\mathbb{R}^{2N+1}}u_0(\eta)d\eta\geq 0.\tag{1.4}$$

In [4] Boutefnouchet and Kirane are established the result of nonexistence solutions for nonlinear nonlocal parabolic system of the type:

$$(NLPS):\begin{cases}\frac{\partial u}{\partial t}-\Delta_{\mathbb{H}}^{\frac{\beta}{2}}(u) & = & \mid v\mid^{p}\\ \frac{\partial v}{\partial t}-\Delta_{\mathbb{H}}^{\frac{\beta}{2}}(v) & = & \mid u\mid^{q}\end{cases}\tag{1.5}$$

2010 *Mathematics Subject Classification.* 35A01; 35B33; 35R03; 35R11; 35R45.

Key words and phrases. Critical exponent; fractional derivative; evolution equation; test function method; Heisenberg group .

where they proved that the system (NLPS) admits no solution defined in $\mathbb{H}^N$ whenever

$$Q \leq \frac{\beta}{pq-1} max\{p, q\} \tag{1.6}$$

In this paper, we generalize this result to system of tow inequalities of the form

$$(FRDS) : \begin{cases} \mathbf{D}^{\alpha}_{0/t} u - \left(\Delta_{\mathbb{H}}^{\frac{\beta}{2}}\right)(u) & \geq \quad | \, v \, |^p \\ \mathbf{D}^{\delta}_{0/t} v - \left(\Delta_{\mathbb{H}}^{\frac{\sigma}{2}}\right)(v) & \geq \quad | \, u \, |^q \end{cases} \tag{1.7}$$

admits no solution in H^N whenever

$$Q < Q^*_e = \frac{\beta}{pq-1} max\{pq(\alpha-1)+\delta p+1, pq(\delta-1)+\alpha q+1\} \tag{1.8}$$

2. Main results

Definition 2.1. *A local weak solution of the system $(FRDS)$ in $Q_T = \mathbb{R}^{2N+1} \times (0,T)$ with positive initial data $u_0, v_0 \in L^1_{loc}(\mathbb{R}^{2N+1})$, is a pair of locally integrable functions (u, v) such that $(u, v) \in L^q(Q_T) \times L^p(Q_T)$ satisfying:*

$$\int_{Q_T} \left(-u D^{\alpha}_{t/T} \varphi + u \left(\Delta_{\mathbb{H}}^{\frac{\beta}{2}}\right)(\varphi) + |v|^p \varphi + u_0(\eta) D^{\alpha}_{t/T} \varphi \right) d\eta dt \leq 0. \tag{2.1}$$

and

$$\int_{Q_T} \left(-v D^{\delta}_{t/T} \varphi + v \left(\Delta_{\mathbb{H}}^{\frac{\beta}{2}}\right)(\varphi) + |u|^q \varphi + v_0(\eta) D^{\delta}_{t/T} \varphi \right) d\eta dt \leq 0. \tag{2.2}$$

for any non-negative test function $\varphi \in C^1\left((0,T]; H^{\beta}(\mathbb{R}^{2N+1})\right) \cap C\left((0,T]; H^{\beta}(\mathbb{R}^{2N+1})\right)$ such that $\varphi(., T) = 0$.

Remark 2.2. *We assume that the integrals in (2.1) and (2.2) are convergent. In Definition 2 if $T = +\infty$, then the solution is called global.*

Theorem 2.3. *Assume that*

$$Q < Q^*_e = \frac{\beta}{pq-1} max\{pq(\alpha-1)+\delta p+1, pq(\delta-1)+\alpha q+1\}$$

Then there is no weak nontrivial solution (u, v) of the system $(FRDS)$.

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General decay of the solution of a one dimensional double porous elastic system with memory

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Abstract In this paper, we consider a one dimensional elastic system with two porous structures and with memory effects in both porous equations. We establish a general decay result depending on the kernels of the memory terms and the wave speeds of the system. Our result improves and generalizes the previous results, in the sense that the energy dissipation of our system is weaker than that produced by porous dampings and thermal effect as in [5, 17] and that the exponential and polynomial rates of decay are only special cases.
Keywords : Double porosity, memory term, stability, general decay.
AMS Classification : 35B35, 35B40, 35Q70, 74D10, 74F10

1 Introduction

In the present work we are concerned by the following double porous elastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x, & \text{in } (0,\infty)\times(0,L), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \int_0^t g(t-s)\,\varphi_{xx}(s)\,ds & \text{in } (0,\infty)\times(0,L), \\ \kappa_2\psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h(t-s)\,\psi_{xx}(s)\,ds & \text{in } (0,\infty)\times(0,L), \end{cases} \tag{1}$$

with the boundary conditions

$$u_x(0,t) = u_x(\pi,t) = \varphi(0,t) = \varphi(\pi,t) = \psi(0,t) = \psi(\pi,t) = 0 \tag{2}$$

and the initial data

$$\begin{gathered} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x). \end{gathered} \tag{3}$$

where, u is the transversal displacement of a one dimensional elastic solid of length π, φ and ψ are the porous unknown functions, the coefficients

$\rho, \kappa_1, \kappa_2, \mu, \alpha, \beta, \gamma, \alpha_1, \alpha_2$ are positive and the matrix

$$A = \begin{pmatrix} \mu & b & d \\ b & \alpha_1 & \alpha_3 \\ d & \alpha_3 & \alpha_2 \end{pmatrix}$$

is positive definite. The functions g, h are relaxation functions that satisfy the following hypothesis

(H1) $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are C^1 non increasing functions satisfying

$$g(0) > 0,\ l = \alpha - \int_0^{+\infty} g(s)\, ds > 0,$$
$$h(0) > 0,\ k = \gamma - \int_0^{+\infty} h(s)\, ds > 0,$$

and

$$lk > \beta^2. \tag{4}$$

(H2) There exist two positive constants η, ξ such that

$$g'(t) \leq -\xi g(t),\ h'(t) \leq -\eta h(t),\ t \geq 0.$$
$$4\tau_1\tau_4 > (\tau_2 + \tau_3)^2.$$

The energy associated with the solution of system (1)

$$E(t) = \frac{1}{2} \int_0^{\pi} \left[\rho u_t^2 + \kappa_1 \varphi_t^2 + \kappa_2 \psi_t^2 + \mu u_x^2 + \alpha_1 \varphi^2 + \alpha_2 \psi^2\right] dx$$
$$+ \frac{1}{2}\left(\alpha - \int_0^t g(s)\, ds\right) \int_0^{\pi} \varphi_x^2 dx + \frac{1}{2}\left(\gamma - \int_0^t h(s)\, ds\right) \int_0^{\pi} \psi_x^2 dx$$
$$+ b \int_0^{\pi} u_x \varphi dx + d \int_0^{\pi} u_x \psi dx + \beta \int_0^{\pi} \varphi_x \psi_x dx + \frac{1}{2}\left[g \circ \varphi_x + h \circ \psi_x\right]$$

where

$$E'(t) = -g(t) \int_0^{\pi} \varphi_x^2(t)\, dx + \frac{1}{2}(g' \circ \varphi_x) - h(t) \int_0^{\pi} \psi_x^2(t)\, dx + \frac{1}{2}(h' \circ \psi_x),$$
$$\leq \frac{1}{2}(g' \circ \varphi_x) + \frac{1}{2}(h' \circ \psi_x) \leq 0$$

Theorem 1 *Assume that the assumptions (H1) and (H2) hold, then for every* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0;\pi) \times L^2(0;\pi)$ *and* $(u_0; u_1) \in H_*^1(0;\pi) \times L_*^2(0;\pi)$; *the problem has a unique weak solution* (u, φ, ψ) *satisfies*

$$u \in C\left((0, +\infty), H_*^1(0, \pi)\right) \cap C^1\left((0, +\infty), L_*^2(0, \pi)\right),$$
$$\varphi, \psi \in C\left((0, +\infty), H_0^1(0, \pi)\right) \cap C^1\left((0, +\infty), L^2(0, \pi)\right),$$

where

$$L_*^2(0, \pi) := \left\{v \in L^2(0, \pi);\ \int_0^{\pi} v(x)\, dx = 0\right\} \text{ and } H_*^1 := H^1(0, \pi) \cap L_*^2(0, \pi)$$

2 General decay

Now we are able to state and prove our main result. Our main result is reads as follow

Theorem 2 *Let* $(u_0;u_1)\in H_*^1 1(0;\pi)L_*^2(0;\pi)$ *and* $(\varphi_0,\varphi_1),(\psi_0,\psi_1)\in H_0^1(0,\pi)\times L^2(0,\pi)$

$$\frac{\mu}{\rho}=\frac{\alpha}{\kappa_1}+\frac{d\beta}{b\kappa_2}=\frac{\gamma}{\kappa_2}+\frac{b\beta}{d\kappa_1}$$

Then for any $t_0>0$ *and any non increasing* C^1*-function* $\chi:\mathbb{R}_+\to\mathbb{R}_+$ *satisfying* $\chi(t)\leq min\ \{\xi(t);\eta(t)\}$ *for all* $t\geq 0$; *there exists a positive constant* σ *such that the energy* $E(t)$ *satisfies*

$$E(t)\leq\sigma e^{-\omega\int_{t_0}^{t}\chi(s)ds},\forall t\geq t_0$$

in order to prove this theorem we define the Lyapunov functional

$$\mathcal{L}(t)=NE(t)+N_1F_1(t)+N_2(F_2(t)+F_3(t))+N_4F_4(t)+F_5(t)$$

after many step, we obtain

$$\mathcal{L}'(t)\leq -E(t)+cg\circ\varphi_x+ch\circ\psi_x$$

Let χ be positive constant such that

$$\chi=\min\{\xi,\eta\},$$

then, multiplying by χ we obtain

$$\chi\mathcal{L}'(t)\leq-\lambda\chi E(t)-cg'\circ\varphi_x-ch'\circ\psi_x\leq-\lambda\chi E(t)-cE'(t)$$

$$\frac{d}{dt}(\chi\mathcal{L}(t)+cE(t))\leq-\lambda\chi E(t)$$

Let $\mathcal{F}(t)=\chi\mathcal{L}(t)+cE(t)$, then
F $(t)\sim E(t)$ and there exists a positive constant ω such that

$$\mathcal{F}'(t)\leq-\omega\chi E(t)\ \ \forall t\geq t_0.$$

An integration over (t_0,t) gives

$$\mathcal{F}(t)\leq\mathcal{F}(t_0)\,e^{-\omega\chi(t-t_0)},\ \forall t\geq t_0.$$

Using the equivalence between $\mathcal{F}$ and E we obtain

$$E(t)\leq Ce^{-\omega\chi(t-t_0)},\ \forall t\geq t_0.$$

which completes the proof of Theorem 2

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Existence and Uniqueness of impulsive Differential Equation of Fractional Order

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Introduction

In this paper, we study the existence and uniqueness of solution's initial value problems for fractional order differential equation, witch can be formulated by the following equation :

$$\begin{array}{c} {}^cD^{\alpha}y(t) = f(t,y), t \in J = [0,T], t \neq t_k \\ \Delta y \mid_{t=t_k} = I_k(y_{t_k}^-), \\ y(0) = y_0 \end{array} \tag{1}$$

where $k = 1,...,m$, $0 < \alpha \leq 1$, ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, such that $\forall t \in J : f(t,0) = 0$, $I_k : \mathbb{R} \longrightarrow \mathbb{R}$, and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $\Delta y \mid_{t=t_k} = y_{t_k}^+ - y_{t_k}^-$, $y_{t_k}^+ = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y_{t_k}^- = \lim\limits_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Differential equations of fractional order is one of the most important in analysis, witch able to be effective tools in study of many problems; resulting for the modelling of various phenomena (science, engineering)(see [[1], [2], [3]].
Impulsive differential equations (for $\alpha \in \mathbb{N}$) was very important in the physical and social science problems.
Really, this result is exists in [see[4]], witch he used the very famous theories of fixed point, where the first based to Banach contraction principle, the second based to Schefer's fixed point theorem, and th third based to the nonlinear alternative of Leray-Schauder type. But in this work, we schow with a new generalization of Banach contraction principle (see [5]), where we broader the space to complete semi-normed space provide with the family of semi norms $p = \{p_{\alpha}, \alpha \in A\}$; $A = \{$ the set of all the compacts $\subset \mathbb{R}\}$ and $p_{\alpha}(u) = \sup\limits_{t \in \alpha}\{e^{-\lambda t}u(t)\}; \lambda \in \mathbb{R}$.

Preliminary

Semi-normed space

Définition 1 *Let E be a K- vector space. We call that p is semi-norm on E when the function $p : E \longrightarrow \mathbb{R}_+$ satisfies the following conditions :*
(i) $\forall x \in E, \forall \lambda \in K : p(\lambda x) = |\lambda| p(x)$.
(ii) $\forall x \in E, \forall y \in E : p(x+y) \leq p(x) + p(y)$.

Définition 2 *The space E is semi-normed space; when it provided with a separate family of semi-norms $P = \{p_{\alpha}, \alpha \in A\}$.*

Generalization of Banach contraction principle in no-metric space

Let E be a complete separate locale convex vector topological space provided with the family of semi-norms $P = \{p_{\alpha} : \alpha \in A\}$, such that A is the set of indices. Let $X \subset E$ a sub-set of E and $T : X \longrightarrow E$ be non-linear function. The application T is non-self mapping.

Théoreme 1 (see [5]) *Let x a j-bounded set and let $T : X \to E$ be a contractive function such that :*

(a) $\sum_{n=0}^{\infty} 2^{2n} K_{\alpha}.K_{j(\alpha)}...K_{j^n(\alpha)} < \infty$ *for all* $\alpha \in A$;
(b) *for any* $x \in \partial X$, $Tx \in X$.
So T has a unique fixed point in X.

Main Result

In this section, we proved that the last equation has a unique solution. For that, we supposed that the conditions (H) are valued :

(H1) There exists a constant $l > 0$ such that $|f(t,u)f(t,\overline{u})| \leq l - *|u - \overline{u}|$, for each $t \in J$, and each $u, \overline{u} \in \mathbb{R}$.

(H2) There exists a constant $l^* > 0$ such that $|I_k(u) - I_k(\overline{u}) \leq l^*|u - \overline{u}|$, for each $u, \overline{u} \in \mathbb{R}$ and $k = 1, ..., m$.

Let $E = C(\mathbb{R})$ provide with the semi-norms family $\mathcal{P} := \{p_K(f), K \in A\}$, such that $p_K(f) = \sup\{e^{-\lambda t}|f(t)| : t \in K\}$. the set$A$ contained all the real compact sub-sets. The function j be introduced by

$$j(K) = \cup_{i=1}^{n} U_i(K); U_i(K) := \{U_i(t), t \in K\}$$

where $U_i(t) = \delta_i(t), 1 \leq \delta_i \leq 2$. We see $K \subseteq j(K)$

Conclusion

In this work, the main but is study the existence and uniqueness of impulsive differential equations of fractional order $\alpha \in [0,1]$, witch be define in semi-norm space E.

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Fractional multipoint boundary value problems at resonance with kernel dimension greater than one

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Abstract

Based on the well-known coincidence degree theory of Mawhin, we have obtained a new existence result for a class of fractional boundary value problems at resonance when the differential operator has kernel 2. An example of application supports our result.

Keywords: Coincidence degree, Fredholm operator, resonance, fractional boundary value problem.

Introduction

In this paper, we are concerned with the following nonlinear fractional differential equation (FDE for short):

$$D_{0^+}^{\alpha} u(t) = f\left(t, u(t), D_{0^+}^{\alpha-2} u(t), D_{0^+}^{\alpha-1} u(t)\right), \quad 0 < t < 1 \tag{1}$$

subject to the boundary conditions

$$u(0) = 0, D_{0^+}^{\alpha-1} u(0) = \sum_{i=1}^{i=m} \beta_i D_{0^+}^{\alpha-1} u(\eta_i), \Gamma(\alpha-1) u(1) = D_{0^+}^{\alpha-2} u\left(\frac{1}{\alpha-1}\right), \tag{2}$$

where $2 < \alpha \leq 3$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a given L^1-Caratheodory function, and $0 < \eta_1 < \eta_2 < \ldots < \eta_m < 1$ are m given points. Finally, the coefficients $\beta_i \in \mathbb{R}$, for $i = 1, 2, 3, \ldots, m$ $(m \geq 2)$ satisfy the resonance condition: $\sum_{i=1}^{i=m} \beta_i = 1$. We will assume that there exists $q \in \mathbb{R}_+$ such that $\sum_{i=1}^{i=m} \beta_i \eta_i^{q+1} = 0$. It is clear that the boundary value problem (1)-(2) is at resonance. Indeed, equation $D_{0^+}^{\alpha} u(t) = 0$ together with the conditions (2)-(**??**) has $u(t) = at^{\alpha-1} + bt^{\alpha-2}$ as nontrivial solutions, where $a, b \in \mathbb{R}$ and $t \in [0,1]$. As a consequence, a direct Green's function cannot be defined. However, taking into account (**??**), we will show that the sum of two linear operators, defined in R1R2, will provide a fixed point formulation to problem fde-bc.

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Résumé :

Le travail étude le convergence d'un solution de système différentiel fractionnaires perturbé au sans de Capito avec des conditions initial (une étude de cas de la perturbation d'ordre de la dérivation fractionnaire et son approche à 1 a gauche et droite ou l'ordre enter 0 et 1)
(p_{ε^-}) :

$$\begin{cases} ({}^cD^{1-\varepsilon}u_\varepsilon)(t) = f(t, u_\varepsilon(t)), & 0 < 1-\varepsilon < 1 \\ u_\varepsilon(0) = u_{\varepsilon,0} & t \in [0,T], \quad T < +\infty \end{cases}$$

(p_{ε^+}) :

$$\begin{cases} ({}^cD^{1+\varepsilon}u_\varepsilon)(t) = f(t, u_\varepsilon(t)), & 0 < \varepsilon < 1 \\ u_\varepsilon(0) = u_{\varepsilon,0} & t \in [0,T], \quad T < +\infty \\ u'_\varepsilon(0) = \varepsilon\mu_0 \end{cases}$$

vers une solution d'un système différentiel du premier ordre
(p_0) :

$$\begin{cases} (Du)(t) = f(t, u(t)), \\ u(0) = u_0, & t \in [0,T] \end{cases}$$

(avec l'aide de la théorème de Gronwell).
et etude le cas ou l'ordre de dérivées de Capito $0 < \alpha_\varepsilon < 1$

Mots clés :

perturbation-théorème de Gronwell-dérivées fractionnaires -Fonction de Mittag-Leffler

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EXTENDED SPECTRUM-PRESERVING ADDITIVE MAPS

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ABSTRACT. Let X be a complex Banach space. We Show that an extended spectrum preserving surjective additive map ϕ in $\mathcal{B}(X)$ is either of the form $\phi(T) = ATA^{-1}$ for linear isomorphism A of X or of the form $\phi(T) = BT^*B^{-1}$ for linear isomorphism B of X^*.

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Key words and phrases. Extended spectrum, extended eigenvalue, surjective additive map.

MULTIPLE SOLUTIONS FOR THE p-FRACIONAL LAPLACIAN WITH CRITICAL GROWTH

Djamel Abid and Kamel Akrout

Abstract. This paper deals with the existence of at least three nontrivial solutions to the following p-fractional Laplacian problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda |u|^{p-2} u + f(x,u) + \mu g(x,u) \text{ in } \Omega, u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n (n > ps)$, is a bounded smooth domain, $s \in (0,1), \lambda,\ \mu$ are positive parameters, and $f, g : \overline{\Omega} \times [0, \infty) \longrightarrow \mathbb{R}$, are continuous functions. Using variational methods and a classical concentration compacteness method.

1. Introduction

In this paper, motivated by the above mentioned works, we consider the following p-fractional Laplacian system

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda |u|^{p-2} u + f(x,u) + \mu g(x,u) \text{ in } \Omega, u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n, n > ps,\ s \in (0,1), \lambda$ and μ are positive parameters, the functions $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_+$, are positive continuous differentiable with respect to the second argument and $f(x,0) = 0, \quad g(x,0) = 0$, satisfaing specific conditions.

Theorem 1.1. *under assuptions (??), (??) , there exist $\mu^* > 0$ depending only on n, p, q and the constant α_3, such that for every $\mu > \mu^*$. there exist three different, nonetrivial, (weak) solutions of problem (1.1). Moreover these solutions are, one negative, one positive and the ather has non-constant sign.*

2. Proof of theorem 1

we will use in this proof the folowing lemmas

Lemma 2.1. *For every $u_0 \in X_0, u_0 > 0, \quad (u_0 < 0)$, there exists $t_\mu > 0$ such that $t_\mu u_0 \in M_1, (t_\mu u_0 \in M_2)$. Moreover, $\lim_{\mu\to\infty} t_\mu = 0$.*

Lemma 2.2. *There exists $c > 0$ such that,*

$$\|u_-\|_{X_0} \geq c \quad \texttt{for} \quad u \in K_2$$
$$\|u_+\|_{X_0} \geq c \quad \texttt{for} \quad u \in K_1$$
$$\|u_-\|_{X_0}, \|u_+\|_{X_0} \geq c \quad \texttt{for} \quad u \in K_3$$

Lemma 2.3. *There exists $c' > 0$ such that $\Phi(u) \geq c' \|u\|_X^p$ for every $u \in X$ if $\|u\|_{X_0}$ is small enough.*

Lemma 2.4. *M_i, is a C^1 sub-manifold of X_0 of co-dimension 1,2 for $i = 1,2$ and of co-dimension 2 for $i = 3$ respectively. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_u X_0 = T_u M_i \bigoplus \mathtt{span}\{u_-, u_+\},$$

where $T_u M$ is the tangent space at u of the banach manifold M. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Lemma 2.5. *The unrestricted fuctional Φ verifies the palais-Smale condition for energy level $c < \frac{1}{n} S_p^{\frac{n}{p}}$. Where S_p is the Sobolev constant given by (**??**).*

Proof. The proof is based on the concentration compactness method See ([8]). □

We will prove the Palais-Smale condition for the function Φ restricted to the manifold M_i.

Lemma 2.6. *The functional Φ defined on K_i satifies the Plais-Smale condition for energy level c for every $c < \frac{1}{n} S_p^{\frac{n}{p}}$.*

Lemma 2.7. *Let $u \in K_i$ be a critical point of the restricted functional $\Phi|_{K_i}$. Then u is also a critical point of the unrestricted funcational Φ and hence a weak solution to problem (1.1).*

3. Proof of Theorem 1

First we need to show that the fuctional $\Phi|_{K_i}$. satifies the hypothesis of the Ekcland's Variational Principle ([6]). We have as a direct consequence of the construction of the manifold K_i that Φ is bounded below over K_i.
Hence, by Ekcland's Variational Principle, there existe $v_k \in K_i$ such that

$$\Phi(v_k) \to c_i := \inf_{K_i} \Phi \quad \textbf{and} \quad (\Phi|_{K_i})'(v_k) \to 0.$$

And we heva frome lemma (2.1) if we choose μ large, $c_i < \frac{1}{n} S_p^{\frac{n}{p}}$. For instace, for c_1, we get the choosing $w_0 \geq 0$,

$$c_1 \leq \Phi(t_\mu) w_0) \leq \frac{1}{p} t_\mu^p A(w_0).$$

Therefore $c_1 \to 0$ as $\mu \to +\infty$. Moreover, it follows from lemma (2.1) that $c_i < \frac{1}{n} S_p^{\frac{n}{p}}$ for $\mu > \mu^*(p, q, n\alpha_3, \beta_3)$.
Frome lemma (2.5), there exists a convergent subsequence extracted from v_k still denoted v_k. Therefore the funcational Φ has a critical point in K_i, $i = 1, 2, 3$.

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Mathematical modeling of nano solar cells base on resolution of partial differential equation system.

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Abstract :

Nano solar cells are mathematically modeled using Poisson's equation and the partial differential equations of charge carrier continuity for both electrons and holes. This system resolution conducts to mathematical solutions that describe the physical behavior of the simulated solar cell and gives an insight into electrical parameters. The simulated solar cell is NIP based structure using the following layers: PCBM/MAPbI3/PEDOT:PSS where PCBM is the electron transporting layer (ETL), the MAPbI3 is the absorbing layer (PAL), and the PEDOT:PSS is the hole transporting layer (HTL). The model is validated by experimental results and is mathematically optimized to find out better power conversion efficiency. Found results show a good optimization in PCE which is very important to experimental applications.

Keywords : MAPbI3, Poisson's equation, differential partial equation resolution, solar cell

Introduction :

Recently, solar cell applications have a huge development [1] to find out new configurations that yields high power conversion efficiency [2]. However, experimental efforts are very expensive, and time consumer [3], which enabled researchers to do simulation efforts to predict experimental behavior using mathematical modeling of physical system [4]. In nano technology applications, especially in solar cell field of research, is used three principal partial differential equations which are Poisson's equation and charge continuity equations for electron and holes for mathematical modeling. In this paper is investigated and modeled a NIP solar cell with three layers : PCBM/MAPbI3/PEDOT:PSS to study its behavior and find out better performances in power conversion efficiency.

Device structure and simulation methodology

Simulations are made based on resolution of Poisson's equation (1) and charge continuity equations for electrons and holes (2,3) :

$$\frac{\partial}{\partial x}\left(\varepsilon(x)\frac{\partial\psi}{\partial x}\right)=-q\left[-n+p-N_A^-+N_D^++\frac{1}{q}\rho_{def}\left(n,p\right)\right] \tag{1}$$

$$-\frac{\partial j_n}{\partial x}+G-U_n(n,p)=\frac{\partial n}{\partial t} \tag{2}$$

$$-\frac{\partial j_p}{\partial x}+G-U_p(n,p)=\frac{\partial p}{\partial t} \tag{3}$$

Where :

p, *n* are free carrier concentration for electron and holes respectively,

N^-_A, N^+_D are dopant charges for acceptors and donors respectively,

$\rho_{def}(n,p)$ is the defect distribution,

J_n, J_p are the electron and hole current densities respectively,

U_n, U_p are the net recombination rates for electron and holes respectively,

G is the generation rate.

The three above equation are resolved with consideration of constitutive relations (4,5) :

$$j_n=-\frac{\mu_n}{q}\frac{\partial E_{Fn}}{\partial x} \tag{4}$$

$$j_p=-\frac{\mu_p}{q}\frac{\partial E_{Fp}}{\partial x} \tag{5}$$

In SCAPS software [5], differential partial equations are resolved for potential and quasi Fermi levels, with boundary conditions at interfaces and contacts, where the structure is discretized, meshing refined around interfaces using Newton-Raphson method with algorithm modifications.

In figure 1 is presented the solar cell structure where is to notice that it contains three layers, the upper one is the PCBM which plays the role of electron transporting layer, the middle one is the MAPbI3 absorbing layer, and lower one is the PEDOT:PSS is the hole transporting layer. The PAL in this structure, when exposed to sun sheen, generates charge carriers where electrons and holes are directed in a way to generate a current flow between the upper and lower contacts.

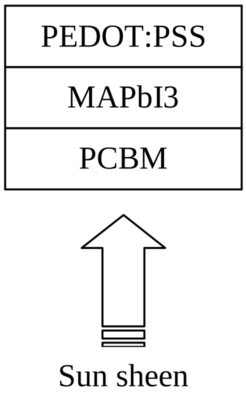


Figure 1. Simulated structure.

Simulation parameters are carefully selected from experimental works [6-10] and are presented in table 1.

Table 1 : electrical and optical parameters for each layer.

	PCBM	**PEDOT:PSS**	**MAPbI3**
Eg **(eV)**	2	1.6	1.55
χ **(eV)**	3.9	3.4	3.75
Nc **(cm^{-3})**	2.5E21	1E22	2.2E15
Nv **(cm^{-3})**	2.5E21	1E22	2.2E17
N^{+}_{D} **(cm^{-3})**	2.93E17	-	1E14
N^{-}_{A} **(cm^{-3})**	-	1E22	5E16
ε_r	3.9	3	6.5
μ_n **(cm^{2}V^{-1}s^{-1})**	0.02	4.5E-4	2
μ_p **(cm^{2}V^{-1}s^{-1})**	0.02	9.9E-5	2
Defect density (cm^{-3})	1E15	2.5E15	1.5E16

Result and discussions

Including various parameters presented in table 1 and according to structure presented in figure 1, differential partial equations (1, 2, 3, 4; 5) are resolved using SCAPS software. Initially, the model is confirmed compared to experimental results where it is noticed a good agreement in JV curves as presented in figure 2. Used layer thickness for simulated model are 30 nm, 30 nm, and 400 nm for ETL, HTL, and PAL respectively. After that, layer thickness of

ETL, HTL, and PAL are tuned one by one to find out the better configuration that gives the maximum of PCE. In each step is taken the thickness that gives better PCE.

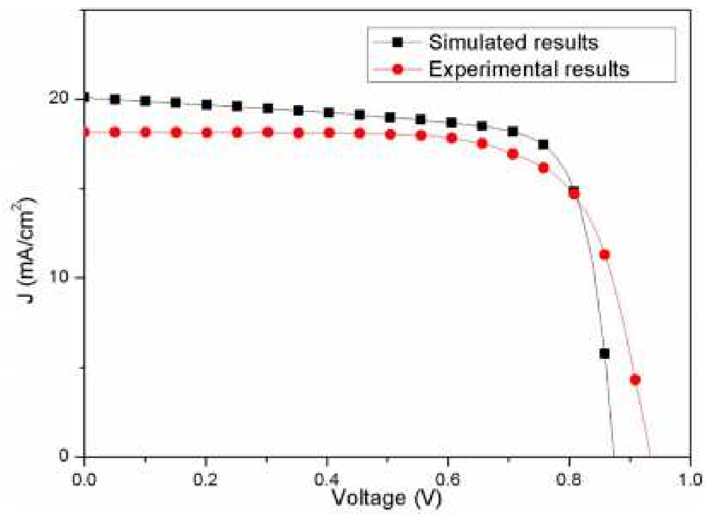
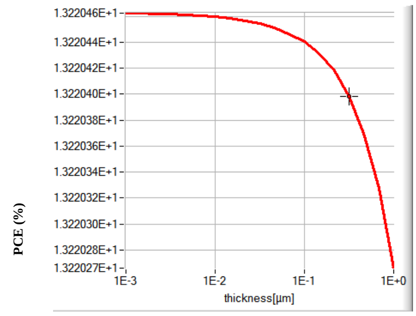
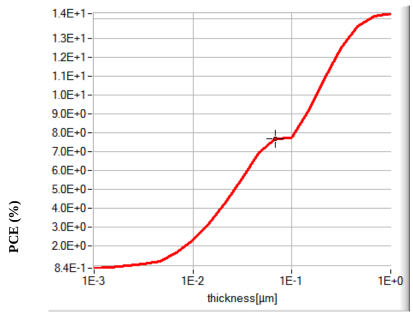


Figure 2. Experimental and simulated JV characteristics.

In figure 3 is presented HTL layer thickness effect on PCE, ETL layer thickness on PCE, and PAL thickness effect on PCE. It is to notice that PAL layer thickness increases the PCE because of the generation of more charge carriers. However, ETL and HTL thickness rduces the PCE because of increasement of recombination rate that decreases charge carrier concentration and hence reduces the PCE. Optimal found layer thickness is 30 nm, 10 nm, 900 nm for HTL, ETL, and PAL layers which gives a PCE value of 15.16 %.

a) b)





c)

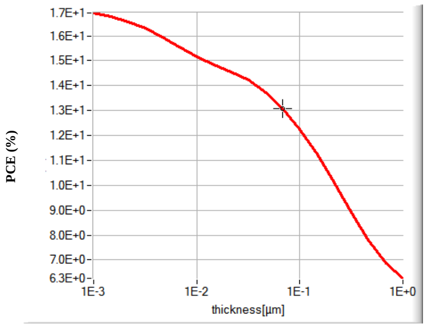


Figure 3. Layer thickness effect on PCE for a) HTL, b) ETL, c) PAL.

Conclusion

In this paper is proofed that mathematical modeling of solar cells is a powerful tool to have physical properties insight prior to experimental efforts, to gain time and money. An NIP solar cell was studied and enhanced based on differential partial equation resolution using Newton-Raphson numerical method. The PCE of the studied solar cell was enhanced from 13.22 % to 15.16% with layer thickness processing. Initial layer thickness was (according to experimental work) 30 nm, 30 nm, and 400 nm for ETL, HTL, and PAL respectively. Optimized layer thickness is found to be 30 nm, 20 nm, and 900 nm for HTL, ETL, and PAL respectively.

We can conclude that mathematical modeling is indispensable to have an idea on behavior of physical phenomena.

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NATIONAL CONFERENCE ON APPLIED MATHEMATICS AND MATHEMATICAL MODELING (A3M'2021)

HAMMA LAKHDAR ELOUED UNIVERSITY

TITLE

EXISTENCE OF POSITIVE SOLUTION FOR A CLASS OF SINGULAR INFINITE SEMIPOSITONE ELLIPTIC SYSTEMS

ZEDIRI SONIA AND AKROUT KAMEL

Abstract. We consider following elliptic system

$$\begin{cases} -\Delta_{pi} u_i = \mu_i u_i^{p_i - 1} - a_i \prod_{j=1}^{m} u_j^{\alpha_{ij}}, i = \overline{1, m}, \text{ on } \Omega. \\ \\ u_i = 0, i = \overline{1, m} \text{ on} \partial\Omega, \end{cases}$$

we discuss the existence of positive solution via sub-super solution, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, a_i $\mu_i, i = \overline{1, m}$ are a positive parameters, and $-1 < \alpha_{ii} < 0, i = \overline{1, m}, 0 < \alpha_{ij} < 1, i \neq, j = \overline{1, m}$.

1. Introduction

In this work, we study the existence of positive solution to infinite semipositone systems of the form

$$(1.1) \qquad \begin{cases} -\Delta_{pi} u_i = \mu_i u_i^{p_i - 1} - a_i \prod_{j=1}^{m} u_j^{\alpha_{ij}}, i = \overline{1, m}, \text{ in } \Omega, \\ \\ u_i = 0, \forall i = \overline{1, m} \text{ on, } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ_{p_i} is the p_i-Laplace operator, and $p_i > 1, a_i, \mu_i > 0, i = \overline{1, m}$,
$-1 < \alpha_{ii} < 0$, $0 < \alpha_{ij} < 1, \forall i, j = \overline{1, m}, i \neq j$ are a positive constants satisfies

$$(1.2) \qquad \sum_{j=1}^{m} \alpha_{ij} < 0, \text{and } \frac{1+\alpha_{ii}}{p_i^*} + \sum_{i \neq j=1}^{m} \frac{\alpha_{ij}}{p_j^*} < 1.$$

Our approach is based on the method of sub and suprsolution were the first eingenfunction is used to construct the sub solution of the problem (1.1).

Date:
Key words and phrases. Positive solutions; Sub-supersolutions; Infinite Semipositone systems.

Lemma 1. *For $p>1$ the following problem*

$$\begin{cases} -\Delta_p u = f(x), \textit{in } \Omega, \\ u = 0, \textit{on } \partial\Omega, \end{cases}$$

has a unique positive solution $u \in W_0^{1,p}(\Omega)$ if $f \in L^{p'}(\Omega)$.

Lemma 2. *If there exist sub-supersolutions $(\psi_i)_{i=\overline{1,m}}$ and $(z_i)_{i=\overline{1,m}}$, respectively, such that $\psi_i \leq z_i, i=\overline{1,m}$ on Ω, then (1.1) has at least a positive solutions satisfying $\psi_i \leq u_i \leq z_i, i=\overline{1,m}.$ on Ω.*

Let $\lambda_i, i=1,m$ be the first eigenvalue of $-\Delta_{pi}, i=\overline{1,m}$ with Dirichlet boundary conditions and φ_i the corresponding positive eigenfunction.

There exists positive constants h_{ij} $i,j=\overline{1,m}$ such that

$$h_{ji}^{-1}\varphi_j \leq \varphi_i \leq h_{ij}\varphi_j,\ i,j=\overline{1,m} \text{ for all } x \in \Omega. \tag{1.3}$$

With $\|\varphi_i\| = 1$, $i=\overline{1,m}$, and $M_i, \varepsilon, \upsilon_i > 0$, such that

$$\begin{aligned} &|\nabla\varphi_i| \geq M_i,\ i=\overline{1,m} \text{ on } \bar{\Omega} = \{x \in \Omega : d(x,\partial\Omega) \leq \varepsilon\}. \\ &\varphi_i \geq \upsilon_i > 0,\ i=\overline{1,m} \text{ on } \Omega \backslash \Omega_\varepsilon. \end{aligned} \tag{1.4}$$

We denote by

$$\begin{cases} A = \left(\overset{\wedge}{\alpha}_{ij}\right)_{i,j=\overline{1,m}}, \overset{\wedge}{\alpha}_{ij} = -\alpha_{ij} + \delta_{ij}(p_i-1), \delta_{ij} = \begin{cases} 1, & i=j. \\ 0, & i \neq j. \end{cases} . \\ \ominus = (\theta_i)_{ij=\overline{1,m}} . \\ P = (p_i)_{i=\overline{1,m}} . \end{cases}$$

Knowing that $\ominus$ is a solution of the algebraic system

$$A\ominus = P, \text{withe } \det(A) \neq 0.$$

Let

$$C = \max_{1 \leq i \leq m}\left(\frac{M_i}{a_i}\left(\prod_{i\neq j=1}^{m} h_{ji}^{-\alpha_{ij}\theta_j}\right)\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\right)^{-\left(\sum_{j=1}^{m}\overset{\wedge}{\alpha}_{ij}\right)^{-1}}.$$

And

$$\mu_i^* = \theta_i^{p_i-1}\left(\lambda_i + \frac{M_i(\theta_i-1)(p_i-1)}{\upsilon_i}\right), i=\overline{1,m}.$$

2. Main results

We consider problem (1.1) under the following assumptions

$$\det(A) \neq 0. \tag{2.1}$$

$$\theta_i > 1, \forall i = \overline{1,m}. \tag{2.2}$$

Theorem 1. *Let* (1.2)-(1.3),(2.1) *and* (2.2) *hold. Then for a positive constant* A *such that*

$$\mu_i^* \leq \mu_i \leq a_i A^{-\sum_{j=1}^{m} \hat{\alpha}_{ij}},\ i = \overline{1,m}, \tag{2.3}$$

the system (1.1) *has a positive solution.*

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